EXISTENCE AND BLOW-UP BEHAVIOR OF STANDING WAVES FOR THE GROSS-PITAEVSKII FUNCTIONAL WITH A HIGHER ORDER INTERACTION

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Abstract. We study the constraint minimization problem related to the Gross-Pitaevskii functional with a higher order interaction

$$I_a^\delta := \inf \left\{ E_a^\delta(\phi) : \phi \in \mathcal{H}(\mathbb{R}^2), \|\phi\|^2_{L^2} = 1 \right\},$$

where $\delta > 0, a > 0$,

$$E_a^\delta(\phi) := \int_{\mathbb{R}^2} |\nabla \phi|^2 dx + \int_{\mathbb{R}^2} V|\phi|^2 dx + \frac{\delta}{2} \int_{\mathbb{R}^2} |\nabla(|\phi|^2)|^2 dx - \frac{1}{2}\int_{\mathbb{R}^2} |\phi|^4 dx$$

and V is a continuous periodic potential. Thanks to the concentration-compactness principle, we show the existence of minimizers for I_a^δ with $a \geq a^* := \|Q\|^2_{L^2}$ and δ sufficiently small, where Q is the unique positive radial solution to

$$-\Delta Q + Q - Q^3 = 0.$$

The blow-up behaviors of minimizers for I_a^δ as $\delta \searrow 0$ are described in details with an additional assumption on the external potential in the case $a = a^*$.

1. Introduction

In this paper, we study the following Gross-Pitaevskii equation with a higher order interaction in two dimensions

$$i\partial_t \Phi + \Delta \Phi = V\Phi - a|\Phi|^2\Phi - \delta\Delta(|\Phi|^2)\Phi, \tag{1.1}$$

where $\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$, $a, \delta > 0$ and $V : \mathbb{R}^2 \to \mathbb{R}$ is an external potential. When $\delta = 0$, the equation (1.1) is the well-known Gross-Pitaevskii equation (GPE) appeared in the theory of Bose-Einstein condensation (BEC). The Bose-Einstein condensates have been investigated intensively since its first realization in cold atomic gases [2, 7]. In the derivation of GPE, one key assumption is that the binary interaction between the particles can be well described by pseudopotential approximation. Such approximation is valid in low energies and becomes less valid in high energies. Therefore, numerous efforts have been devoted to the improvements of the pseudopotential approximation for the two-body interaction, which lead to better mean field theory towards the understanding of BEC experiments. In [4, 8], a higher order interaction correction to the pseudopotential approximation has been proposed and analyzed. As a consequence, a BEC with a higher order interaction can be described by the wave function whose evolution is governed by the dimensionaless modified Gross-Pitaevskii equation (1.1) (see also [23, 24]).

We are interested in standing waves solutions for (1.1), i.e. solutions of the form $\Phi(t, x) = e^{i\omega t}\phi(x)$, where $\omega \in \mathbb{R}$ is a frequency, and $\phi \in \mathcal{H}(\mathbb{R}^2)$ is a non-trivial solution to the elliptic equation

$$-\Delta\phi + \omega\phi + V\phi - \delta\Delta(|\phi|^2)\phi - a|\phi|^2\phi = 0, \tag{1.2}$$

where $\mathcal{H}(\mathbb{R}^2)$ is a subspace of $H^1(\mathbb{R}^2)$ is defined by

$$\mathcal{H}(\mathbb{R}^2) := \left\{ \phi \in H^1(\mathbb{R}^2) \ : \ \|\nabla(|\phi|^2)\|^2_{L^2} < \infty \right\}.$$

The existence of standing waves solutions for (1.1) can be achieved by several ways. The first way is to minimize the functional

$$J(u) := \int |\nabla\phi|^2 dx + \omega \int |\phi|^2 dx \int V|\phi|^2 dx + \frac{\delta}{2}\int |\nabla(|\phi|^2)|^2 dx$$

on the set $\{u \in \mathcal{H} \ : \ \|u\|^4_{L^4} = 1\}$ (see e.g. [18, 21]). The second way is to use a change of variables to transform (1.2) into a related semilinear problem which allows one to apply the mountain pass theorem and other variational methods (see e.g. [5, 17]). Note that since the term $\|\nabla(|\phi|^2)\|^2_{L^2}$ is not convex and $\mathcal{H}(\mathbb{R}^2)$

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is not a vector space, the usual min-max techniques cannot be directly applied. Another approach is to look for critical points of the energy functional

$$E_a^\delta(\phi) := \|\nabla \phi\|_{L^2}^2 + A(\phi) + \frac{\delta}{2}\|\nabla(|\phi|^2)\|_{L^2}^2 - \frac{a}{2}\|\phi\|_{L^4}^4$$

with prescribed L^2-norm, where

$$A(\phi) := \int V|\phi|^2 dx. \tag{1.3}$$

In the later case, the parameter ω is no longer given, but rather a corresponding Lagrange multiplier of the constrained minimization problem.

Motivated by the fact that physicists are often interested in normalized solutions to (1.2), we consider the minimizing problem: for $a>0$ and $\delta>0$,

$$I_a^\delta := \inf\{E_a^\delta(\phi) \ : \ \phi \in \mathcal{H}(\mathbb{R}^2), \|\phi\|_{L^2}^2 = 1\}. \tag{1.4}$$

In the case $\delta = 0$, the existence, non-existence and blow-up behavior of minimizers for I_a^0 have been studied in many works. In the case of trapping potential, i.e.

$$V \in L^\infty_{\text{loc}}(\mathbb{R}^2), \quad \inf_{x\in\mathbb{R}^2} V(x) = 0, \quad \lim_{|x|\to\infty} V(x) = \infty, \tag{1.5}$$

Guo-Seiringer [9] proved that for any $0<a<a^*$, there exists at least a minimizer for I_a^0; and for any $a \geq a^*$, there is no minimizer for I_a^0. Here

$$a^* := \|Q\|_{L^2}^2 \tag{1.6}$$

with Q the unique positive radial solution to

$$-\Delta Q + Q - Q^3 = 0. \tag{1.7}$$

They also gave a detailed description of the blow-up behavior of minimizers for I_a^0 as $a \nearrow a^*$. The result of Guo-Seiringer was then extended to ring shaped potentials, i.e.

$$V(x) = (|x|-A)^2, \quad A>0$$

by Guo-Zeng-Zhou [10], to continuous periodic potentials satisfying

$$0 = \min V < \inf \sigma(-\Delta + V)$$

by Wang-Zhao [25], where

$$\inf \sigma(-\Delta+V) := \inf\left\{\|\nabla\phi\|_{L^2}^2 + A(\phi) \ : \ \phi \in H^1(\mathbb{R}^2), \|\phi\|_{L^2}^2 = 1\right\},$$

to singular potentials by Phan [19], to multi-well potentials by Guo-Wang-Zeng-Zhou [11], to ellipse-shaped potentials by Guo-Zhou [13] and to rotating trap potentials by Guo-Luo-Yang [12].

When $\delta>0$, the existence of minimizer for I_a^δ with no external potential was studied by Colin-Jeanjean-Squassina in [6]. In the case of trapping potential, Bao-Cai-Ruan [3] proved the existence of minimizers for I_a^δ for any $a>0$ and any $\delta>0$ (see also [26] for higher dimensions). They also studied the asymptotic behavior of minimizers for I_a^δ with $a>a^*$ as $\delta \searrow 0$ by assuming

$$V(x) = \sum_{i=1}^2 \nu_i x_i^2, \quad \nu_i > 0, x=(x_1,x_2)\in\mathbb{R}^2. \tag{1.8}$$

More precisely, they proved (see [3, Theorem 3.4]) that if ϕ_δ is a non-negative minimizer for I_a^δ with $a > a*$, then up to a subsequence

$$\sqrt{\delta}\phi_\delta(\sqrt{\delta}\cdot) \to \phi_0 \text{ strongly in } H^1(\mathbb{R}^2), \tag{1.9}$$

where ϕ_0 is a non-negative minimizer for

$$d_a := \inf\left\{F_a(\phi) \ : \ \phi\in\mathcal{H}(\mathbb{R}^2), \|\phi\|_{L^2}^2 = 1\right\} \tag{1.10}$$

with

$$F_a(\phi) := \|\nabla\phi\|_{L^2}^2 + \frac{1}{2}\|\nabla(|\phi|^2)\|_{L^2}^2 - \frac{a}{2}\|\phi\|_{L^4}^4.$$

Note that for any $a>a^*$, there exists at least a minimizer for d_a (see e.g. [27, Lemma 1]). The proof of (1.9) is based on some technical rescaling arguments which use the homogeneity of the potential V defined in (1.8). Recently, Zeng-Zhang [27] extended the result of Bao-Cai-Ruan by considering a more general trapping potential. More precisely, they proved that if $V \in C^{\alpha}_{\rm loc}(\mathbb{R}^2)$ satisfies (1.5) and ϕ_δ is a non-negative minimizer for I^δ_a with $a \geq a^*$, then there exists $(x_\delta)_{\delta\searrow 0} \subset \mathbb{R}^2$ satisfying

$$\lim_{\delta\searrow 0} x_\delta = x_0 \in \mathbb{R}^2, \quad V(x_0) = 0 \tag{1.11}$$

such that

- if $a = a^*$, then
$$\varepsilon_\delta \phi_\delta(\varepsilon_\delta \cdot + x_\delta) \to Q_0 \text{ strongly in } H^1(\mathbb{R}^2)$$
with $\varepsilon_\delta \to 0$ as $\delta \searrow 0$, where
$$Q_0 := \frac{Q}{\|Q\|_{L^2}}, \quad \varepsilon_\delta := \|\nabla \phi_\delta\|^{-1}_{L^2}; \tag{1.12}$$
- if $a > a^*$, then
$$\sqrt{\delta}\phi_\delta(\sqrt{\delta} \cdot + x_\delta) \to \phi_0 \text{ strongly in } \mathcal{H}(\mathbb{R}^2),$$
where ϕ_0 is a non-negative minimizer for d_a. Moreover,
$$\lim_{\delta\searrow 0}\left(I^\delta_a - \frac{d_a}{\delta}\right) = 0.$$

In addition, under a suitable assumption of the trapping potential, they gave a precise blow-up rate of $\|\nabla\phi_\delta\|_{L^2}$ with $a = a^*$ as $\delta \searrow 0$. More precisely, they assumed that V has l different minimal points and there exist r_0 and $p_i, \gamma_i > 0, i = 1, \cdots, l$ such that

$$V(x) = \gamma_i |x - x_i|^{p_i} + o(|x - x_i|^{p_i}), \quad \forall x \in B_{r_0}(x_i).$$

Let

$$p := \max\{p_i \ : \ i = 1, \cdots, l\}, \quad \gamma := \min\{\gamma_i \ : \ p_i = p\}$$

and

$$\mathcal{Z} := \{x_i \ : \ p_i = p, \quad \gamma_i = \gamma\}.$$

They proved that the blow-up point x_0 defined in (1.11) belongs to $\mathcal{Z}$ and

$$\varepsilon_\delta \approx \left(\frac{2\lambda_1}{\gamma\lambda_2 p}\delta\right)^{\frac{1}{4+p}} \text{ as } \delta \searrow 0,$$

where

$$\lambda_1 := \int_{\mathbb{R}^2} |\nabla(Q_0^2)(x)|^2 dx, \quad \lambda_2 := \int_{\mathbb{R}^2} |x|^p [Q_0(x)]^2 dx. \tag{1.13}$$

Here $A_\delta \approx B_\delta$ as $\delta \searrow 0$ means that $\lim_{\delta\searrow 0} \frac{A_\delta}{B_\delta} = 1$.

The main purpose of this paper is to extend the result of Bao-Cai-Ruan [3] and Zeng-Zhang [27] to a class of continuous periodic potentials, i.e.

(A1) $V \in C(\mathbb{R}^2), \quad V(x + z) = V(x)$ for all $x \in \mathbb{R}^2, z \in \mathbb{Z}^2$.

Our first result is the existence of minimizers for I^δ_a with $a \geq a^*$ and δ sufficiently small.

Theorem 1.1. *Let V satisfy* (A1)*. Then it holds that*

- *If $a > a^*$, then there exists $\delta_* > 0$ such that for any $0 < \delta < \delta_*$, there exists at least a minimizer for I^δ_a;*
- *If $a = a^*$ and assume in addition that*
  (A2) $\min V < \inf \sigma(-\Delta + V)$,
  then there exists $\delta_* > 0$ such that for any $0 < \delta < \delta_*$, there exists at least a minimizer for $I^\delta_{a^*}$. Moreover,
$$\lim_{\delta\searrow 0} I^\delta_{a^*} = I^0_{a^*} = \min V.$$

Comparing to the case of no external potential or trapping potentials, the existence of minimizers for I^δ_a is more involved. In the case of no external potential, one may take advantage of the Schwarz rearrangement to show the existence of minimizers for I^δ_a (see [6]). In the case of trapping potentials, it is well-known that the space

$$\mathcal{K}(\mathbb{R}^2) := \left\{\phi \in H^1(\mathbb{R}^2) \ : \ A(\phi) < \infty\right\}$$

is compactly embedded into $L^q(\mathbb{R}^2)$ for any $2 \leq q < \infty$ (see (1.3) for the defintion of $A(\phi)$). This compact embedding allows one to obtain easily the existence of minimizers for I^δ_a (see e.g. [3, 26]). In the case of continuous periodic potentials, due to the lack of compactness at infinity, we will use a suitable version of the concentration-compactness principle to show the existence of minimizers for I^δ_a.

Remark 1.2. In the case of no higher order interaction, i.e. $\delta = 0$, the existence of minimizer for I_a^0 with continuous periodic potentials was established by Wang-Zhao [25]. They proved that under the assumptions (A1) and (A2), there exists $0 < a_* < a^*$ such that for any $a_* < a < a^*$, there exists at least a minimizer for I_a^0; and for any $a \geq a^*$, there is no minimizer for I_a^0. In the case of a higher order interaction, i.e. $\delta > 0$, the existence of minimizer for I_a^δ is completely different as Theorem 1.1 indicates.

Remark 1.3. We also have from Remark 2.3 that for any $\delta > 0$ fixed, there exists $a_* > 0$ such that for any $a > a_*$, there exists at least a minimizer for I_a^δ.

We are next interested in the blow-up behavior of minimizers for I_a^δ with $a \geq a^*$ as $\delta \searrow 0$. Note that we can restrict our consideration to non-negative minimizers for I_a^δ since $E_a^\delta(|\phi|) \leq E_a^\delta(\phi)$ for all $\phi \in \mathcal{H}(\mathbb{R}^2)$.

Theorem 1.4. *Let ϕ_δ be a non-negative minimizer for I_a^δ given in Theorem 1.1. Then there exists $(x_\delta)_{\delta\searrow 0} \subset [0,1]^2$ satisfying*

$$x_\delta \to x^0 \in [0,1]^2, \quad V(x^0) = \min V$$

and $(z_\delta)_{\delta\searrow 0} \subset \mathbb{Z}^2$ such that

- *if $a = a^*$, then*
$$\varepsilon_\delta \phi_\delta(\varepsilon_\delta \cdot + x_\delta + z_\delta) \to Q_0 \text{ strongly in } H^1(\mathbb{R}^2),$$
where Q_0 and ε_δ are defined in (1.12);
- *if $a > a^*$, then*
$$\sqrt{\delta}\phi_\delta(\sqrt{\delta} \cdot + x_\delta + z_\delta) \to \phi_0 \text{ strongly in } \mathcal{H}(\mathbb{R}^2),$$
where ϕ_0 is a non-negative minimizer for d_a. Moreover,
$$\lim_{\delta\searrow 0} \left(I_a^\delta - \frac{d_a}{\delta} \right) = \min V.$$

In the case $a > a^*$, $\|\nabla \phi_\delta\|_{L^2}$ blows up at speed $1/\sqrt{\delta}$ as $\delta \searrow 0$. This blow-up rate is independent of the external potential. In the case $a = a^*$, Theorem 1.4 does not give any information on the blow-up rate of $\|\nabla\phi_\delta\|_{L^2}$ as $\delta \searrow 0$ due to the lack of information on the local behavior of the external potential around its minimal points. Inspired by the work of Wang-Zhao [25], we make the following assumption

(A3) $\min V = 0, V^{-1}(0) = x_0 + \mathbb{Z}^2$ for some $x_0 \in [0,1]^2$ and there exist $\gamma, p > 0$ such that
$$\lim_{x\to x_0} \frac{V(x)}{|x-x_0|^p} = \gamma.$$

Theorem 1.5. *Let $a = a^*$ and V satisfy (A1), (A2) and (A3). Let ϕ_δ be a non-negative minimizer for $I_{a^*}^\delta$ given in Theorem 1.1. Then*

$$x^0 \equiv x_0, \quad \lim_{\delta\searrow 0} \frac{x_\delta - x_0}{\varepsilon_\delta} = 0, \quad \varepsilon_\delta \approx \left(\frac{2\lambda_1}{\gamma\lambda_2 p}\delta \right)^{\frac{1}{4+p}} \text{ as } \delta \searrow 0, \tag{1.14}$$

where x_δ and x^0 are as in Theorem 1.4, and λ_1, λ_2 are defined in (1.13). Moreover,

$$\lim_{\delta\searrow 0} \delta^{-\frac{p}{4+p}} I_{a^*}^\delta = (4+p) \left(\frac{\lambda_1}{2p} \right)^{\frac{p}{4+p}} \left(\frac{\gamma\lambda_2}{4} \right)^{\frac{4}{4+p}}. \tag{1.15}$$

Theorem 1.5 gives a detailed description of the blow-up behavior of minimizers ϕ_δ for $I_{a^*}^\delta$ as $\delta \searrow 0$. This implies that $\|\nabla\phi_\delta\|_{L^2}$ blows up at speed $\left(\frac{2\lambda_1}{\gamma\lambda_2 p}\delta\right)^{-\frac{1}{4+p}}$ as $\delta \searrow 0$. Moreover, as $\delta \searrow 0$, a minimizer ϕ_δ for $I_{a^*}^\delta$ behaves like

$$\phi_\delta(x) \approx \left(\frac{2\lambda_1}{\gamma\lambda_2 p}\delta \right)^{-\frac{1}{4+p}} Q_0 \left(\left(\frac{2\lambda_1}{\gamma\lambda_2 p}\delta \right)^{-\frac{1}{4+p}} (x - x_\delta - z_\delta) \right).$$

The paper is organized as follows. In Section 2, we give the proof of the existence of minimizers for I_a^δ given in Theorem 1.1. The blow-up behavior of minimizers for $I_{a^*}^\delta$ as $\delta \searrow 0$ is proved in Section 3.

2. Existence of minimizers

We first recall the following concentration-compactness principle due to Colin-Jeanjean-Squassina [6] which is a modified version of the concentration-compactness principle of Lions [15, 16].

Lemma 2.1 (Concentraion-compactness principle [6]). *Let $N \geq 1$ and $(\phi_n)_{n\geq 1}$ be a bounded sequence in $\mathcal{H}(\mathbb{R}^N)$, i.e. there exists $C > 0$ independent of n such that*

$$\|\phi_n\|_{L^2}^2 + \|\nabla \phi_n\|_{L^2}^2 + \|\nabla(|\phi_n|^2)\|_{L^2}^2 \leq C, \quad \forall n \geq 1.$$

Asumme that

$$\lim_{n\to\infty} \|\phi_n\|_{L^2}^2 = c$$

for some fixed costant $c > 0$. Then there exists a subsequence $(\phi_{n_k})_{k\geq 1}$ satisfying one of the following three possibilities:

- **Vanishing.**
$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |\phi_{n_k}(x)|^2 dx = 0$$
for all $R > 0$;
- **Compactness.** *There exists a sequence $(y_k)_{k\geq 1} \subset \mathbb{R}^N$ such that for all $\epsilon > 0$, there exists $R(\epsilon) > 0$ and $k(\epsilon) \geq 1$ such that for all $k \geq k(\epsilon)$,*
$$\int_{B(y_k,R(\epsilon))} |\phi_{n_k}(x)|^2 dx \geq c - \epsilon;$$
- **Dichotomy.** *There exist $\alpha \in (0, c)$ and bounded sequences $(\phi_k^1)_{k\geq 1}, (\phi_k^2)_{k\geq 1} \subset \mathcal{H}(\mathbb{R}^N)$ such that*
$$\begin{cases} \lim_{k\to\infty} \|\phi_{n_k} - \phi_k^1 - \phi_k^2\|_{L^q} = 0 \text{ for any } 2 \leq q < \overline{2}, \\ \lim_{k\to\infty} \|\phi_k^1\|_{L^2}^2 = \alpha, \quad \lim_{k\to\infty} \|\phi_k^2\|_{L^2}^2 = c - \alpha, \\ \lim_{k\to\infty} \mathrm{dist}(\mathrm{supp}(\phi_k^1), \mathrm{supp}(\phi_k^2)) = \infty, \\ \liminf_{k\to\infty} \|\nabla \phi_{n_k}\|_{L^2}^2 - \|\nabla \phi_k^1\|_{L^2}^2 - \|\nabla \phi_k^2\|_{L^2}^2 \geq 0, \\ \liminf_{k\to\infty} \|\nabla(|\phi_{n_k}|^2)\|_{L^2}^2 - \|\nabla(|\phi_k^1|^2)\|_{L^2}^2 - \|\nabla(|\phi_k^2|^2)\|_{L^2}^2 \geq 0, \end{cases} \tag{2.1}$$
where $\overline{2} := \frac{2(N+2)}{N-2}$ if $N \geq 3$ and $\overline{2} := \infty$ if $N = 1, 2$.

We refer the reader to [6, Section 5] for the proof of this result. The essential differeces between this version and the classical one of Lions are the last condition in the dichotomy and the exponent $\overline{2}$ which is greater than the classical exponent $2^* = \frac{2N}{N-2}$ if $N \geq 3$.

Remark 2.2. We also have from [6, Section 5] that

- if the vanishing occurs, then $\phi_{n_k} \to 0$ strongly in $L^q(\mathbb{R}^N)$ for any $2 < q < \overline{2}$;
- if the compactness occurs, then up to a subsequence, $\phi_{n_k}(\cdot + y_k)$ converges to some ϕ weakly in $\mathcal{H}(\mathbb{R}^2)$ and strongly in $L^q(\mathbb{R}^N)$ for any $2 \leq q < \overline{2}$.

We also need the following Nash inequality (see [1, 14, 26])

$$\int_{\mathbb{R}^N} |f|^{\frac{q+2}{2}} dx \leq C_{\mathrm{opt}} \left(\int_{\mathbb{R}^N} |\nabla f|^2 dx \right)^{\frac{(q+2)\theta_q}{4}} \left(\int_{\mathbb{R}^N} |f| dx \right)^{\frac{(q+2)(1-\theta_q)}{2}}, \quad \forall f \in \mathcal{D}^{2,1}(\mathbb{R}^N), \tag{2.2}$$

where

$$1 < \frac{q+2}{2} < \frac{2N}{N-2} \text{ if } N \geq 3, \quad 1 < \frac{q+2}{2} < \infty \text{ if } N = 1, 2, \quad \theta_q = \frac{2qN}{(q+2)(N+2)}$$

and

$$\mathcal{D}^{2,1}(\mathbb{R}^N) := \left\{ f \ : \ \nabla f \in L^2(\mathbb{R}^N), f \in L^1(\mathbb{R}^N) \right\}.$$

It was proved in [1] that the optimal constant $C_{\mathrm{opt}} = \frac{1}{\lambda_q a_q}$ with

$$\lambda_q := (1 - \theta_q) \left(\frac{\theta_q}{1 - \theta_q} \right)^{\frac{qN}{2(N+2)}}, \quad a_q := \|R_q\|_{L^1}^{\frac{q}{N+2}},$$

where R_q optimizes (2.2) and is the unique non-negative radially symmetric solution to the elliptic equation

$$-\Delta R_q + 1 - R_q^{\frac{q}{2}} = 0, \quad x \in \mathbb{R}^N.$$

It is known (see [22]) that R_q has a compact support in $\mathbb{R}^N$ and exactly satisfies a Dirichlet-Neumann free boundary problem, that is, there exists one $R > 0$ such that R_q is the unique solution to

$$\begin{cases} -\Delta f + 1 - f^{\frac{q}{2}} = 0, \\ f > 0 \text{ on } B_R, \quad f = \frac{\partial f}{\partial n} = 0 \text{ on } \partial B_R. \end{cases}$$

We can rewrite (2.2) as

$$\|\phi\|^{q+2}_{L^{q+2}} \leq C_{\text{opt}} \|\nabla(|\phi|^2)\|_{L^2}^{\frac{qN}{N+2}} \|\phi\|_{L^2}^{\frac{2(N+2)-(N-2)q}{N+2}}, \quad \forall \phi \in \mathcal{H}(\mathbb{R}^N). \tag{2.3}$$

Such an inequality is available for $0 < q < \overline{2}$. The sharp constant C_{opt} becomes

$$C_{\text{opt}} = \frac{1}{\lambda_q \|W_q\|_{L^2}^{\frac{2q}{N+2}}},$$

where W_q optimizes (2.3) and is the unique non-negative solution to

$$-\Delta(W_q^2) + 1 - W_q^q = 0, \quad x \in \mathbb{R}^N.$$

We also have the following Pohozaev identities for W_q

$$\|W_q\|^{q+2}_{L^{q+2}} = \frac{1}{1-\theta_q}\|W_q\|^2_{L^2}, \quad \|\nabla(W_q^2)\|^2_{L^2} = \frac{\theta_q}{1-\theta_q}\|W_q\|^2_{L^2}. \tag{2.4}$$

In the case $N = 2$ and $q = 2$, we have the following Nash inequality

$$\|\phi\|^4_{L^4} \leq C_{\text{opt}} \|\nabla(|\phi|^2)\|_{L^2} \|\phi\|^2_{L^2}, \quad \forall \phi \in \mathcal{H}(\mathbb{R}^2), \tag{2.5}$$

where the sharp constant

$$C_{\text{opt}} = \frac{2}{\|W\|_{L^2}}.$$

Here $W := W_2$ optimizes (2.5) and is the unique non-negative solution to

$$-\Delta(W^2) + 1 - W^2 = 0. \tag{2.6}$$

We are now able to prove the existence of minimizers for I_a^δ.

Proof of Theorem 1.1. The proof is done by several steps.

Step 1. I_a^δ is well-defined for any $a > 0$ and $\delta > 0$. Indeed, let $\phi \in \mathcal{H}(\mathbb{R}^2)$ be such that $\|\phi\|^2_{L^2} = 1$. By the Nash inequality (2.5) and Young inequality, we have for any $\epsilon > 0$,

$$\|\phi\|^4_{L^4} \leq C\|\nabla(|\phi|^2)\|_{L^2}\|\phi\|^2_{L^2} \leq \epsilon\|\nabla(|\phi|^2)\|^2_{L^2} + C\epsilon^{-1}\|\phi\|^4_{L^2} = \epsilon\|\nabla(|\phi|^2)\|^2_{L^2} + C\epsilon^{-1}.$$

It follows that

$$E_a^\delta(\phi) \geq \|\nabla\phi\|^2_{L^2} + \min V + \frac{1}{2}(\delta - a\epsilon)\|\nabla(|\phi|^2)\|^2_{L^2} - Ca\epsilon^{-1}. \tag{2.7}$$

By choosing $\epsilon > 0$ small enough, we infer that $I_a^\delta > -\infty$.

Step 2. For $a \geq a^*$ fixed, there exists $\delta_* > 0$ such that $I_a^\delta < \inf \sigma(-\Delta + V)$ for all $0 < \delta < \delta_*$. To see this, let $x^0 \in \mathbb{R}^2$ and Q be the unique positive radial solution to (1.7). Consider the trial function

$$\phi_\tau(x) := \tau Q_0(\tau(x - x^0)), \quad \tau > 0,$$

where $Q_0 = \frac{Q}{\|Q\|_{L^2}}$. It is easy to see that

$$\|\phi_\tau\|^2_{L^2} = 1, \quad \|\nabla\phi_\tau\|^2_{L^2} = \tau^2\|\nabla Q_0\|^2_{L^2}, \quad \|\phi_\tau\|^4_{L^4} = \tau^2\|Q_0\|^4_{L^4}, \quad \|\nabla(|\phi_\tau|^2)\|^2_{L^2} = \tau^4\|\nabla(Q_0^2)\|^2_{L^2}.$$

Note that $Q_0 \in \mathcal{H}(\mathbb{R}^2)$ since Q_0 and $|\nabla Q_0|$ are bounded and decay exponentially at infinity. Moroever,

$$A(\phi_\tau) = \int V(x)\tau^2[Q_0(\tau(x - x^0))]^2 dx = \int V(\tau^{-1}x + x^0)[Q_0(x)]^2 dx \to V(x^0)$$

as $\tau \to \infty$ due to the dominated convergence theorem and the fact V is uniformly bounded on $\mathbb{R}^2$. We thus get

$$\begin{aligned} E_a^\delta(\phi_\tau) &= \tau^2\|\nabla Q_0\|^2_{L^2} + V(x^0) + \frac{\delta}{2}\tau^4\|\nabla(Q_0^2)\|^2_{L^2} - \frac{a}{2}\tau^2\|Q_0\|^4_{L^4} + o_\tau(1) \\ &= \left(1 - \frac{a}{a^*}\right)\tau^2 + \frac{\delta}{2}\tau^4\|\nabla(Q_0^2)\|^2_{L^2} + V(x^0) + o_\tau(1) \end{aligned}$$

as $\tau \to \infty$, where $A_\tau = o_\tau(1)$ means $A_\tau \to 0$ as $\tau \to \infty$. Here we have used the fact

$$\|\nabla Q\|^2_{L^2} = \|Q\|^2_{L^2} = \frac{1}{2}\|Q\|^4_{L^4} \text{ or } \|\nabla Q_0\|^2_{L^2} = 1, \quad \|Q_0\|^4_{L^4} = \frac{2}{\|Q\|^2_{L^2}} = \frac{2}{a^*}$$

which follows from Pohozaev identities for (1.7).

If $a > a^*$, then we choose $\tau > 0$ such that

$$\delta\tau^4\|\nabla(Q_0^2)\|^2_{L^2} = \left(\frac{a}{a^*} - 1\right)\tau^2 \text{ or } \tau^2 = \left(\frac{a}{a^*} - 1\right)\frac{1}{\delta\|\nabla(Q_0^2)\|^2_{L^2}}.$$

Note that $\tau \to \infty$ as $\delta \searrow 0$. It follows that

$$E_a^\delta(\phi_\tau) = -\frac{1}{2}\left(\frac{a}{a^*}-1\right)^2 \frac{1}{\delta\|\nabla(Q_0^2)\|_{L^2}^2} + V(x^0) + o_{\delta\searrow 0}(1).$$

This implies that $I_a^\delta \to -\infty$ as $\delta \searrow 0$. There thus exists $\delta_* > 0$ such that $I_a^\delta < \inf\sigma(-\Delta+V)$ for all $0<\delta<\delta_*$.

If $a = a^*$, we have

$$I_{a^*}^\delta \le \frac{\delta}{2}\tau^4\|\nabla(Q_0^2)\|_{L^2}^2 + V(x^0) + o_\tau(1).$$

Choosing $\tau = \delta^{-1/8}$, we see that

$$I_{a^*}^\delta \le \frac{1}{2}\sqrt{\delta}\|\nabla(Q_0^2)\|_{L^2}^2 + V(x^0) + o_{\delta\searrow 0}(1)$$

Letting $\delta \to 0$ and optimizing the right hand side, we get

$$\limsup_{\delta\searrow 0} I_{a^*}^\delta \le \min V. \tag{2.8}$$

On the other hand, by the Gagliardo-Nirenber inequality, we have for any $\phi \in \mathcal{H}(\mathbb{R}^2)$ satisfying $\|\phi\|_{L^2}^2 = 1$,

$$E_{a^*}^\delta(\phi) \ge E_{a^*}^0(\phi) \ge A(\phi)$$

which implies that $I_{a^*}^\delta \ge I_{a^*}^0 \ge \min V$ for any $\delta > 0$. It yields that

$$\liminf_{\delta\searrow 0} I_{a^*}^\delta \ge I_{a^*}^0 \ge \min V$$

which together with (2.8) imply

$$\lim_{\delta\searrow 0} I_{a^*}^\delta = I_{a^*}^0 = \min V. \tag{2.9}$$

By (A2), there exists $\delta_* > 0$ such that $I_{a^*}^\delta < \inf\sigma(-\Delta+V)$ for all $0<\delta<\delta_*$.

Step 3. Existence of minimizers for I_a^δ. Let $(\phi_n)_{n\ge1}$ be a minimizing sequence for I_a^δ. By (2.7), $(\phi_n)_{n\ge1}$ is a bounded sequence in $\mathcal{H}(\mathbb{R}^2)$. By the concentration-compactness principle, there exists a subsequence $(\phi_{n_k})_{k\ge1}$ satisfying one of the three possibilities: vanishing, dichotomy and compactness.

No vanishing. If the vanishing occurs, then $\phi_{n_k} \to 0$ strongly in $L^q(\mathbb{R}^2)$ for any $2<q<\infty$. In particular, $\|\phi_{n_k}\|_{L^4}^4 \to 0$ as $k\to\infty$. It follows that

$$I_a^\delta = \lim_{k\to\infty} E_a^\delta(\phi_{n_k}) = \lim_{k\to\infty} \|\nabla\phi_{n_k}\|_{L^2}^2 + A(\phi_{n_k}) + \frac{\delta}{2}\|\nabla(|\phi_{n_k}|^2)\|_{L^2}^2 \ge \inf\sigma(-\Delta+V)$$

which contradicts Step 2.

No dichotomy. If the dichotomy occurs, then there exist $\alpha\in(0,1)$ and $(\phi_k^1)_{k\ge1}, (\phi_k^2)_{k\ge1}$ bounded in $\mathcal{H}(\mathbb{R}^2)$ such that (2.1) holds with $c=1$. Since $\|\phi_{n_k} - \phi_k^1 - \phi_k^2\|_{L^4} \to 0$ as $k\to\infty$ and ϕ_k^1, ϕ_k^2 have disjoint supports for k large, we infer that

$$\|\phi_{n_k}\|_{L^4}^4 = \|\phi_k^1\|_{L^4}^4 + \|\phi_k^2\|_{L^4}^4 + o_k(1).$$

Since V is uniformly bounded on $\mathbb{R}^2$ and $\|\phi_{n_k} - \phi_k^1 - \phi_k^2\|_{L^2} \to 0$ as $k\to\infty$, we see that

$$A(\phi_{n_k}) = A(\phi_k^1) + A(\phi_k^2) + o_k(1).$$

We thus show that

$$E_a^\delta(\phi_{n_k}) \ge E_a^\delta(\phi_k^1) + E_a^\delta(\phi_k^2) + o_k(1). \tag{2.10}$$

We next observe that for any $\lambda > 0$,

$$\begin{aligned} E_a^\delta(\lambda\phi) &= \lambda^2\|\nabla\phi\|_{L^2}^2 + \lambda^2 A(\phi) + \frac{\delta}{2}\lambda^4\|\nabla(|\phi|^2)\|_{L^2}^2 - \frac{a}{2}\lambda^4\|\phi\|_{L^4}^4 \\ &= \lambda^4 E_a^\delta(\phi) + \lambda^2(1-\lambda^2)\left(\|\nabla\phi\|_{L^2}^2 + A(\phi)\right) \end{aligned}$$

which implies that

$$E_a^\delta(\phi) = \frac{1}{\lambda^4}E_a^\delta(\lambda\phi) + \left(1-\frac{1}{\lambda^2}\right)\left(\|\nabla\phi\|_{L^2}^2 + A(\phi)\right).$$

Set $\lambda_k^1 := \frac{1}{\|\phi_k^1\|_{L^2}} \ge 1$ and $\lambda_k^2 := \frac{1}{\|\phi_k^2\|_{L^2}} \ge 1$. We have that

$$\begin{aligned} E_a^\delta(\phi_k^1) &\ge \frac{1}{[\lambda_k^1]^4}E_a^\delta(\lambda_k^1\phi_k^1) + \left(1-\frac{1}{[\lambda_k^1]^2}\right)\left(\|\nabla\phi_k^1\|_{L^2}^2 + A(\phi_k^1)\right) \\ &\ge \|\phi_k^1\|_{L^2}^4 I_a^\delta + \left(1-\|\phi_k^1\|_{L^2}^2\right)\|\phi_k^1\|_{L^2}^2 \inf\sigma(-\Delta+V) \\ &= \alpha^2 I_a^\delta + (1-\alpha)\alpha\inf\sigma(-\Delta+V) + o_k(1). \end{aligned}$$

Similarly,

$$E_a^\delta(\phi_k^2) \geq (1-\alpha)^2 I_a^\delta + (1-\alpha)\alpha \inf \sigma(-\Delta+V) + o_k(1).$$

We thus infer from (2.10) that

$$E_a^\delta(\phi_{n_k}) \geq \left(\alpha^2 + (1-\alpha)^2\right) I_a^\delta + 2\alpha(1-\alpha)\inf\sigma(-\Delta+V) + o_k(1).$$

Letting $k\to\infty$, we obtain

$$I_a^\delta \geq \left(\alpha^2+(1-\alpha)^2\right) I_a^\delta + 2\alpha(1-\alpha)\inf\sigma(-\Delta+V)$$

or

$$2\alpha(1-\alpha)I_a^\delta \geq 2\alpha(1-\alpha)\inf\sigma(-\Delta+V)$$

which implies that $I_a^\delta \geq \inf\sigma(-\Delta+V)$ since $\alpha\in(0,1)$. This however contradicts Step 2.

Compactness. Therefore, the compactness must occur. There thus exists $(y_k)_{k\geq 1}\subset\mathbb{R}^2$ such that up to a subsequence $\phi_{n_k}(\cdot+y_k)$ converges to some ϕ weakly in $\mathcal{H}(\mathbb{R}^2)$ and strongly in $L^q(\mathbb{R}^2)$ for any $2\leq q<\infty$. It follows that

$$\begin{aligned}
\|\phi\|_{L^2}^2 &= \lim_{k\to\infty}\|\phi_{n_k}(\cdot+y_k)\|_{L^2}^2 = 1,\\
\|\phi\|_{L^4}^4 &= \lim_{k\to\infty}\|\phi_{n_k}(\cdot+y_k)\|_{L^4}^4 = \lim_{k\to\infty}\|\phi_{n_k}\|_{L^4}^4,\\
\|\nabla\phi\|_{L^2}^2 &\leq \liminf_{k\to\infty}\|\nabla\phi_{n_k}(\cdot+y_k)\|_{L^2}^2 = \lim_{k\to\infty}\|\nabla\phi_{n_k}\|_{L^2}^2
\end{aligned}$$

and by [6, Lemma 4.3],

$$\|\nabla(|\phi|^2)\|_{L^2}^2 \leq \liminf_{k\to\infty}\|\nabla(|\phi_{n_k}(\cdot+y_k)|^2)\|_{L^2}^2 = \lim_{k\to\infty}\|\nabla(|\phi_{n_k}|^2)\|_{L^2}^2.$$

On the other hand,

$$\begin{aligned}
\int V(x)|\phi_{n_k}(x)|^2dx &= \int V(x+y_k)|\phi_{n_k}(x+y_k)|^2dx\\
&= \int V(x+y_k)|\phi(x)|^2dx + \int V(x+y_k)\left(|\phi_{n_k}(x+y_k)|^2-|\phi(x)|^2\right)dx
\end{aligned}$$

and

$$\left|\int V(x+y_k)\left(|\phi_{n_k}(x+y_k)|^2-|\phi(x)|^2\right)dx\right| \leq \|V\|_{L^\infty}\|\phi_{n_k}(\cdot+y_k)-\phi\|_{L^2}\||\phi_{n_k}(\cdot+y_k)|+|\phi|\|_{L^2}\to 0$$

as $k\to\infty$. It follows that

$$\begin{aligned}
E_a^\delta(\phi_{n_k}) &= \|\nabla\phi_{n_k}\|_{L^2}^2 + A(\phi_{n_k}) + \frac{\delta}{2}\|\nabla(|\phi_{n_k}|^2)\|_{L^2}^2 - \frac{a}{2}\|\phi_{n_k}\|_{L^4}^4\\
&\geq \|\nabla\phi\|_{L^2}^2 + \int V(x+y_k)|\phi(x)|^2dx + \frac{\delta}{2}\|\nabla(|\phi|^2)\|_{L^2}^2 - \frac{a}{2}\|\phi\|_{L^4}^4 + o_k(1).
\end{aligned}$$

Since V is periodic, we write $y_k = x_k+z_k$ with $x_k\in[0,1]^2$ and $z_k\in\mathbb{Z}^2$. Since $x_k\in[0,1]^2$, up to a subsequence, $x_k\to x^0\in[0,1]^2$. It follows that

$$\int V(x+y_k)|\phi(x)|^2dx = \int V(x+x_k)|\phi(x)|^2dx \to \int V(x+x^0)|\phi(x)|^2dx$$

as $k\to\infty$ by the dominated convergence since V is uniformly bounded on $\mathbb{R}^2$. We thus get

$$E_a^\delta(\phi_{n_k}) \geq E_a^\delta(\phi(\cdot - x^0)) + o_k(1)$$

which implies that

$$I_a^\delta \geq E_a^\delta(\phi(\cdot-x^0)).$$

Since $\|\phi\|_{L^2}^2=1$, it yields that $I_a^\delta = E_a^\delta(\phi(\cdot-x^0))$ or $\phi(\cdot-x^0)$ is a minimizer for I_a^δ. ☐

Remark 2.3. The argument presented above can be applied to show the existence of minimizers for I_a^δ with $\delta>0$ fixed and a sufficiently large. Indeed, it suffices to show that for $\delta>0$ fixed, there exists $a_*>0$ such that

$$I_a^\delta < \inf\sigma(-\Delta+V), \quad \forall a>a_*. \tag{2.11}$$

To see this, let $x^0\in\mathbb{R}^2$ and W be the unique non-negative solution to (2.6). Define

$$\phi_\tau(x) := \tau W_0(\tau(x-x^0)), \quad \tau>0, \quad W_0 = \frac{W}{\|W\|_{L^2}}.$$

We see that

$$\|\phi_\tau\|_{L^2}^2 = 1, \quad \|\nabla(|\phi_\tau|^2)\|_{L^2}^2 = \tau^4\|\nabla(W_0^2)\|_{L^2}^2, \quad \|\phi_\tau\|_{L^4}^4 = \tau^2\|W_0\|_{L^4}^4, \quad \|\nabla\phi_\tau\|_{L^2}^2 = \tau^2\|\nabla W_0\|_{L^2}^2.$$

On the other hand,

$$A(\phi_\tau) = \int V(x)\tau^2[W_0(\tau(x-x^0))]^2dx = \int V(\tau^{-1}x+x^0)[W_0(x)]^2dx \to V(x^0)$$

as $\tau \to \infty$ due to the dominated convergence theorem. It follows that

$$E_a^\delta(\phi_\tau) = \tau^2\|\nabla W_0\|_{L^2}^2 + V(x^0) + \frac{\delta}{2}\tau^4\|\nabla(W_0^2)\|_{L^2}^2 - \frac{a}{2}\tau^2\|W_0\|_{L^4}^4 + o_\tau(1).$$

By (2.4),

$$\|\nabla(W^2)\|_{L^2}^2 = \|W\|_{L^2}^2 = \frac{1}{2}\|W\|_{L^4}^4$$

which implies that

$$\|\nabla(W_0^2)\|_{L^2}^2 = 1, \quad \|W_0\|_{L^2}^4 = \frac{2}{\|W\|_{L^2}^2}.$$

Thus

$$E_a^\delta(\phi_\tau) = \tau^2\|\nabla W_0\|_{L^2}^2 + V(x^0) + \frac{\delta}{2}\tau^4 - \frac{a}{\|W\|_{L^2}^2}\tau^2 + o_\tau(1).$$

For $\delta > 0$ fixed, we choose $\tau > 0$ such that

$$\delta\tau^4 = \frac{a}{\|W\|_{L^2}^2}\tau^2 \text{ or } \tau^2 = \frac{a}{\delta\|W\|_{L^2}^2}$$

and get

$$E_a^\delta(\phi_\tau) = \frac{a}{\delta\|W\|_{L^2}^2}\|\nabla W_0\|_{L^2}^2 + V(x^0) - \frac{1}{2}\frac{a^2}{\delta\|W\|_{L^4}^4} + o_a(1).$$

This implies that $I_a^\delta \to -\infty$ as $a \to \infty$. There thus exist $a_* > 0$ such that (2.11) holds.

3. Blow-up behavior of minimizers

In this section, we prove the blow-up behavior of minimizers for I_a^δ with $a \geq a^*$ as $\delta \searrow 0$.

Proof of Theorem 1.4. We will consider separately two cases: $a = a^*$ and $a > a^*$.

The case $a = a^*$ Let ϕ_δ be a non-negative minimizer for $I_{a^*}^\delta$ for $0 < \delta < \delta_*$. Set

$$\varepsilon_\delta := \|\nabla\phi_\delta\|_{L^2}^{-1}.$$

We first claim that $\varepsilon_\delta \to 0$ as $\delta \to 0$. In fact, if it is not true, then $(\phi_\delta)_{\delta\searrow 0}$ is a bounded sequence in $H^1(\mathbb{R}^2)$. By the concentration-compactness principle of Lions [15, 16], there exists a subsequence still denoted by $(\phi_\delta)_{\delta\searrow 0}$ such that one of the three possibility holds: vanishing, dichotomy and compactness.

If the vanishing occurs, then $\phi_\delta \to 0$ strongly in $L^q(\mathbb{R}^2)$ for any $2 < q < \infty$. It follows that

$$\min V = I_{a^*}^0 = \lim_{\delta\searrow 0} I_{a^*}^\delta = \lim_{\delta\searrow 0} E_{a^*}^\delta(\phi_\delta) = \lim_{\delta\searrow 0} \|\nabla\phi_\delta\|_{L^2}^2 + A(\phi_\delta) + \frac{\delta}{2}\|\nabla(|\phi_\delta|^2)\|_{L^2}^2 \geq \inf\sigma(-\Delta+V)$$

which contradicts (A2).

If the dichotomy occurs, then there exist $\alpha \in (0,1)$ and bounded sequences $(\phi_\delta^1)_{\delta\searrow 0}$, $(\phi_\delta^2)_{\delta\searrow 0}$ in $H^1(\mathbb{R}^2)$ such that

$$\begin{cases} \lim_{\delta\searrow 0}\|\phi_\delta - \phi_\delta^1 - \phi_\delta^2\|_{L^q} = 0 \text{ for any } 2 \leq q < \infty, \\ \lim_{\delta\searrow 0}\|\phi_\delta^1\|_{L^2}^2 = \alpha, \quad \lim_{\delta\searrow 0}\|\phi_\delta^2\|_{L^2}^2 = 1-\alpha, \\ \lim_{\delta\searrow 0}\operatorname{dist}(\operatorname{supp}(\phi_\delta^1), \operatorname{supp}(\phi_\delta^2)) = \infty, \\ \liminf_{\delta\searrow 0}\|\nabla\phi_\delta\|_{L^2}^2 - \|\nabla\phi_\delta^1\|_{L^2}^2 - \|\nabla\phi_\delta^2\|_{L^2}^2 \geq 0. \end{cases}$$

It follows that

$$E_{a^*}^0(\phi_\delta) \geq E_{a^*}^0(\phi_\delta^1) + E_{a^*}^0(\phi_\delta^2) + o_{\delta\searrow 0}(1).$$

Set $\lambda_\delta^1 := \frac{1}{\|\phi_\delta^1\|_{L^2}} \geq 1$ and $\lambda_\delta^2 := \frac{1}{\|\phi_\delta^2\|_{L^2}} \geq 1$. We have that

$$\begin{aligned} E_{a^*}^0(\phi_\delta^1) &= \frac{1}{[\lambda_\delta^1]^4}E_{a^*}^0(\lambda_\delta^1\phi_\delta^1) + \left(1 - \frac{1}{[\lambda_\delta^1]^2}\right)\left(\|\nabla\phi_\delta^1\|_{L^2}^2 + A(\phi_\delta^1)\right) \\ &\geq \|\phi_\delta^1\|_{L^2}^4 I_{a^*}^0 + \left(1 - \|\phi_\delta^1\|_{L^2}^2\right)\|\phi_\delta^1\|_{L^2}^2 \inf\sigma(-\Delta+V) \\ &= \alpha^2 I_{a^*}^0 + \alpha(1-\alpha)\inf\sigma(-\Delta+V) + o_{\delta\searrow 0}(1). \end{aligned}$$

Similarly,

$$E_{a^*}^0(\phi_\delta^2) \geq (1-\alpha)^2 I_{a^*}^0 + \alpha(1-\alpha)\inf\sigma(-\Delta+V) + o_{\delta\searrow 0}(1).$$

It follows that

$$\begin{aligned} E^\delta_{a^*}(\phi_\delta) \geq E^0_{a^*}(\phi_\delta) &\geq E^0_{a^*}(\phi^1_\delta) + E^0_{a^*}(\phi^2_\delta) + o_{\delta\searrow 0}(1) \\ &\geq \left(\alpha^2+(1-\alpha)^2\right) I^0_{a^*} + 2\alpha(1-\alpha)\inf\sigma(-\Delta+V) + o_{\delta\searrow 0}(1). \end{aligned}$$

Since $E^\delta_{a^*}(\phi_\delta) = I^\delta_{a^*} \to I^0_{a^*} = \min V$ as $\delta \searrow 0$, the above inequality yields that

$$\min V = I^0_{a^*} \geq \left(\alpha^2+(1-\alpha)^2\right) I^0_{a^*} + 2\alpha(1-\alpha)\inf\sigma(-\Delta+V)$$

hence

$$\min V \geq \inf\sigma(-\Delta+V)$$

which again contradicts (A2).

Therefore, the compactness must occurs. By the same argument as in the proof of Theorem 1.1, there exist $\phi \in H^1(\mathbb{R}^2)$ and $x^0 \in \mathbb{R}^2$ such that $\phi(\cdot - x^0)$ is a minimizer for $I^0_{a^*}$. This however contradicts to the fact that $I^0_{a^*}$ has no minimizer (see [25]).

We now set

$$\psi_\delta(x) := \varepsilon_\delta \phi_\delta(\varepsilon_\delta x). \tag{3.1}$$

We see that

$$\|\psi_\delta\|^2_{L^2} = 1, \quad \|\nabla\psi_\delta\|^2_{L^2} = \varepsilon^2_\delta \|\nabla\phi_\delta\|^2_{L^2} = 1.$$

By the Gagliardo-Nirenber inequality and (2.9),

$$\begin{aligned} 0 \leq \|\nabla\psi_\delta\|^2_{L^2} - \frac{a^*}{2}\|\psi_\delta\|^4_{L^4} &= \varepsilon^2_\delta\left(\|\nabla\phi_\delta\|^2_{L^2} - \frac{a^*}{2}\|\phi_\delta\|^4_{L^4}\right) \\ &\leq \varepsilon^2_\delta\left(E^\delta_{a^*}(\phi_\delta) - A(\phi_\delta)\right) \\ &\leq \varepsilon^2_\delta\left(I^\delta_{a^*} - \min V\right) \to 0 \end{aligned}$$

as $\delta \searrow 0$. This implies that

$$\|\psi_\delta\|^2_{L^1} = 1, \quad \|\nabla\psi_\delta\|^2_{L^2} = 1, \quad \lim_{\delta\searrow 0} \|\nabla\psi_\delta\|^2_{L^2} - \frac{a^*}{2}\|\psi_\delta\|^4_{L^4} = 0.$$

By the compactness of optimizing sequence for the Gagliardo-Nirenberg inequality (see e.g. [20, Lemma 4]), there exists a sequence $(y_\delta)_{\delta\searrow 0} \subset \mathbb{R}^2$ such that

$$\psi_\delta(\cdot + y_\delta) \to Q_0 \text{ strongly in } H^1(\mathbb{R}^2). \tag{3.2}$$

We next write $\varepsilon_\delta y_\delta = x_\delta + z_\delta$ with $x_\delta \in [0,1]^2$ and $z_\delta \in \mathbb{Z}^2$. Passing to a subsequence if necessary, $x_\delta \to x^0 \in [0,1]^2$ as $\delta \searrow 0$. It remains to show that $V(x^0) = \min V$. By the Gagliardo-Nirenberg inequality,

$$\min V \leq \int V(x)|\phi_\delta(x)|^2 dx \leq I^\delta_{a^*} \to I^0_{a^*} = \min V$$

as $\delta \searrow 0$. We infer that

$$\begin{aligned} \min V = \lim_{\delta\searrow 0}\int V(x)|\phi_\delta(x)|^2 dx &= \lim_{\delta\searrow 0}\int V(\varepsilon_\delta x)|\psi_\delta(x)|^2 dx \\ &= \lim_{\delta\searrow 0}\int V(\varepsilon_\delta x + \varepsilon_\delta y_\delta)|\psi_\delta(x+y_\delta)|^2 dx \\ &= \lim_{\delta\searrow 0}\int V(\varepsilon_\delta + x_\delta)|\psi_\delta(x+y_\delta)|^2 dx \\ &\geq \int \lim_{\delta\searrow 0} V(\varepsilon_\delta + x_\delta)|\psi_\delta(x+y_\delta)|^2 dx \\ &= V(x^0)\int [Q_0(x)]^2 dx = V(x^0) \end{aligned} \tag{3.3}$$

which implies that $V(x^0) = \min V$. Here we have used the fact that up to a subsequence, $\psi_\delta(\cdot + y_\delta) \to Q_0$ almost everywhere on $\mathbb{R}^2$. We have proved that there exist $(x_\delta)_{\delta\searrow 0} \subset [0,1]^2$ and $(z_\delta)_{\delta\searrow 0} \in \mathbb{Z}^2$ such that

$$x_\delta \to x^0 \in [0,1]^2, \quad V(x^0) = \min V$$

and

$$\varepsilon_\delta\phi_\delta(\varepsilon_\delta \cdot + x_\delta + z_\delta) \to Q_0 \text{ strongly in } H^1(\mathbb{R}^2)$$

as $\delta \searrow 0$.

The case $a > a^*$. Let us first consider

$$d^\delta_a := \inf\left\{F^\delta_a(\phi) \ : \ \phi \in \mathcal{H}(\mathbb{R}^2), \|\phi\|^2_{L^2} = 1\right\},$$

where

$$F_a^\delta(\phi) := \|\nabla\phi\|_{L^2}^2 + \frac{\delta}{2}\|\nabla(|\phi|^2)\|_{L^2}^2 - \frac{a}{2}\|\phi\|_{L^4}^4.$$

Claim 3.1. *Let $a > a^*$ and $\delta > 0$. Then it holds that*

$$\delta d_a^\delta = d_a. \tag{3.4}$$

In particular, ϕ_δ is a minimizer for d_a^δ if and only if $\sqrt{\delta}\phi_\delta(\sqrt{\delta}\cdot)$ is a minimizer for d_a. Recall that d_a is defined in (1.10).

Proof. Let $\phi \in \mathcal{H}(\mathbb{R}^2)$ be such that $\|\phi\|_{L^2}^2 = 1$. Set $\phi_\delta(x) := \sqrt{\delta}\phi(\sqrt{\delta}x)$. It follows that

$$\begin{aligned} F_a(\phi_\delta) &= \|\nabla\phi_\delta\|_{L^2}^2 + \frac{1}{2}\|\nabla(|\phi_\delta|^2)\|_{L^2}^2 - \frac{a}{2}\|\phi_\delta\|_{L^4}^4 \\ &= \delta\left(\|\nabla\phi\|_{L^2}^2 + \frac{\delta}{2}\|\nabla(|\phi|^2)\|_{L^2}^2 - \frac{a}{2}\|\phi\|_{L^4}^4\right) = \delta F_a^\delta(\phi). \end{aligned}$$

Taking the infimum over all $\phi \in \mathcal{H}(\mathbb{R}^2)$ satisfying $\|\phi\|_{L^2}^2 = 1$, we get (3.4). □

Claim 3.2. *Let $a > a^*$. It holds that*

$$\lim_{\delta \searrow 0}\left(I_a^\delta - \frac{d_a}{\delta}\right) = \min V. \tag{3.5}$$

Moreover, if ϕ_δ is a minimizer for I_a^δ, then

$$\lim_{\delta\searrow 0} A(\phi_\delta) = \min V.$$

Proof. Let ϕ_0 be a non-negative radially symmetric minimizer for d_a. Note that such a minimizer for d_a exists (see [27, Lemma 1]). Let $x^0 \in \mathbb{R}^2$ be such that $V(x^0) = \min V$. We consider

$$\phi_0^\delta(x) := \frac{1}{\sqrt{\delta}}\phi_0\left(\frac{1}{\sqrt{\delta}}(x - x^0)\right).$$

Note that by Claim 3.1, ϕ_0^δ is a minimizer for d_a^δ. It is easy to see that

$$\|\phi_0^\delta\|_{L^2}^2 = 1, \quad \|\nabla\phi_0^\delta\|_{L^2}^2 = \frac{1}{\delta}\|\nabla\phi_0\|_{L^2}^2, \quad \|\phi_0^\delta\|_{L^4}^4 = \frac{1}{\delta}\|\phi_0\|_{L^4}^4, \quad \|\nabla(|\phi_0^\delta|^2)\|_{L^2}^2 = \frac{1}{\delta^2}\|\nabla(|\phi_0|^2)\|_{L^2}^2$$

and

$$A(\phi_0^\delta) = \int V(x)\frac{1}{\delta}\left[\phi_0\left(\frac{1}{\sqrt{\delta}}(x - x^0)\right)\right]^2 dx = \int V(\sqrt{\delta}x + x^0)[\phi_0(x)]^2 dx \to V(x^0) = \min V$$

as $\delta \searrow 0$ due to the dominated convergence theorem. We thus get

$$I_a^\delta \le E_a^\delta(\phi_0^\delta) = F_a^\delta(\phi_0^\delta) + A(\phi_0^\delta) = d_a^\delta + \min V + o_{\delta\searrow 0}(1).$$

By Claim 3.1, we infer that

$$\limsup_{\delta\searrow 0}\left(I_a^\delta - \frac{d_a}{\delta}\right) \le \min V. \tag{3.6}$$

On the other hand, we have for any $\phi \in \mathcal{H}(\mathbb{R}^2)$ satisfying $\|\phi\|_{L^2}^2 = 1$,

$$E_a^\delta(\phi) = F_a^\delta(\phi) + A(\phi) \ge F_a^\delta(\phi) + \min V \ge d_a^\delta + \min V.$$

Taking the infimum over all $\phi \in \mathcal{H}(\mathbb{R}^2)$ satisfying $\|\phi\|_{L^2}^2 = 1$, we obtain

$$\liminf_{\delta\searrow 0}\left(I_a^\delta - \frac{d_a}{\delta}\right) \ge \min V$$

which together with (3.6) show (3.5).

Now let ϕ_δ be a minimizer for I_a^δ. It follows that

$$\min V \le A(\phi_\delta) \le E_a^\delta(\phi_\delta) - F_a^\delta(\phi_\delta) = I_a^\delta - F_a^\delta(\phi_\delta) \le I_a^\delta - \frac{d_a}{\delta} \to \min V$$

as $\delta \searrow 0$. The proof is complete. □

Now let ϕ_δ be a non-negative minimizer for I_a^δ given in Theorem 1.1. Set

$$\psi_\delta(x) := \sqrt{\delta}\phi_\delta(\sqrt{\delta}x).$$

We see that

$$\frac{d_a}{\delta} = d_a^\delta \leq F_a^\delta(\phi_\delta) = E_a^\delta(\phi_\delta) - A(\phi_\delta) \leq I_a^\delta - \min V$$

which implies

$$d_a \leq \delta F_a^\delta(\phi_\delta) = F_a(\psi_\delta) \leq \delta I_a^\delta - \delta \min V.$$

By (3.5), we infer that

$$\lim_{\delta \searrow 0} F_a(\psi_\delta) = d_a$$

or $(\psi_\delta)_{\delta\searrow 0}$ is a minimizing sequence for d_a. By [6, Section 5], there exists a sequence $(y_\delta)_{\delta\searrow 0} \subset \mathbb{R}^2$ such that

$$\psi_\delta(\cdot + y_\delta) \to \phi_0 \text{ strongly in } \mathcal{H}(\mathbb{R}^2)$$

as $\delta \searrow 0$ for some ϕ_0 a non-negative minimizer for d_a. We next write $\sqrt{\delta}y_\delta = x_\delta + z_\delta$ with $x_\delta \in [0,1]^2$ and $z_\delta \in \mathbb{Z}^2$. Up to a subsequence, we have $x_\delta \to x^0 \in [0,1]^2$. Moreover, by Claim 3.2,

$$\begin{aligned}
\min V = \lim_{\delta\searrow 0} \int V(x)|\phi_\delta(x)|^2 dx &= \lim_{\delta\searrow 0} \int V(\sqrt{\delta}x)|\psi_\delta(x)|^2 dx \\
&= \lim_{\delta\searrow 0} \int V(\sqrt{\delta}x + \sqrt{\delta}y_\delta)|\psi_\delta(x+y_\delta)|^2 dx \\
&= \lim_{\delta\searrow 0} \int V(\sqrt{\delta}x + x_\delta)|\psi_\delta(x+y_\delta)|^2 dx \\
&\geq \int \lim_{\delta\searrow 0} V(\sqrt{\delta}x + x_\delta)|\psi_\delta(x+y_\delta)|^2 dx \\
&= V(x^0) \int [\phi_0(x)]^2 dx = V(x^0)
\end{aligned}$$

which implies that $V(x^0) = \min V$. We have proved that there exist $(x_\delta)_{\delta\searrow 0} \subset [0,1]^2$ and $(z_\delta)_{\delta\searrow 0} \in \mathbb{Z}^2$ such that

$$x_\delta \to x^0 \in [0,1]^2, \quad V(x^0) = \min V$$

and

$$\sqrt{\delta}\phi_\delta(\sqrt{\delta}\cdot + x_\delta + z_\delta) \to \phi_0 \text{ strongly in } \mathcal{H}(\mathbb{R}^2)$$

as $\delta \searrow 0$. The proof of Theorem 1.4 is now complete. □

We now study the blow-up behavior of minimizers for $I_{a^*}^\delta$ as $\delta \searrow 0$.

Proof of Theorem 1.5. Recall that the assumption (A3) is assumed here.

Claim 3.3. *There exists $C > 0$ independent of δ such that for δ close to 0,*

$$I_{a^*}^\delta \leq C\delta^{\frac{p}{4+p}}. \tag{3.7}$$

Moreover,

$$\limsup_{\delta\searrow 0} \delta^{-\frac{p}{4+p}} I_{a^*}^\delta \leq \inf_{\mu>0} \left(\frac{\lambda_1}{2}\mu^4 + \gamma\lambda_2\mu^{-p}\right), \tag{3.8}$$

where λ_1 and λ_2 are defined in (1.13).

Proof. By (A3), we see that for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$V(x) \leq (\gamma+\epsilon)|x - x_0|^p$$

for all $x \in \mathbb{R}^2$ satisfying $|x - x_0| \leq 2R_\epsilon$. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for all $|x| \leq R_\epsilon$ and $\varphi(x) = 0$ for all $|x| \geq 2R_\epsilon$. Consider a test function

$$\phi_\tau(x) := A_\tau \tau \varphi(x - x_0) Q_0(\tau(x - x_0)), \quad \tau > 0,$$

where $A_\tau > 0$ is such that $\|\phi_\tau\|_{L^2}^2 = 1$ for all $\tau > 0$. Since Q_0 and $|\nabla Q_0|$ are bounded and decay exponentially at infinity, we infer that

$$A_\tau^2 = 1 + O(\tau^{-\infty})$$

as $\tau \to \infty$, where $B_\tau = O(\tau^{-\infty})$ means that for any $\nu > 0$, there exists $C_\nu > 0$ independent of τ such that $B_\tau \le C_\nu \tau^{-\nu}$. We also have that

$$\|\nabla \phi_\tau\|_{L^2}^2 = \tau^2 \|\nabla Q_0\|_{L^2}^2 + O(\tau^{-\infty}),$$
$$\|\phi_\tau\|_{L^4}^4 = \tau^2 \|Q_0\|_{L^4}^4 + O(\tau^{-\infty}),$$
$$\|\nabla(|\phi_\tau|^2)\|_{L^2}^2 = \tau^4 \|\nabla(Q_0^2)\|_{L^2}^2 + O(\tau^{-\infty})$$

as $\tau \to \infty$. Since $0 \le \varphi \le 1$ and $V(x) \le (\gamma+\epsilon)|x-x_0|^p$ on the support of $\varphi(\cdot - x_0)$,

$$\begin{aligned} A(\phi_\tau) &= A_\tau^2 \tau^2 \int V(x)[\varphi(x-x_0)Q_0(\tau(x-x_0))]^2 dx \\ &\le A_\tau^2 \tau^2 (\gamma+\epsilon) \int_{|x-x_0|\le 2R_\epsilon} |x-x_0|^p [Q_0(\tau(x-x_0))]^2 dx \\ &= A_\tau^2 \tau^{-p} (\gamma+\epsilon) \int_{|x|\le 2\tau R_\epsilon} |x|^p [Q_0(x)]^2 dx \\ &= (\gamma+\epsilon)\tau^{-p} \int |x|^p [Q_0(x)]^2 dx + O(\tau^{-\infty}) \end{aligned}$$

as $\tau \to \infty$. Thus

$$I_{a^*}^\delta \le E_{a^*}^\delta(\phi_\tau) = \frac{\delta}{2}\tau^4 \|\nabla(Q_0^2)\|_{L^2}^2 + (\gamma+\epsilon)\tau^{-p} \int |x|^p [Q_0(x)]^2 dx + O(\tau^{-\infty})$$

as $\tau \to \infty$, where we have used the fact that

$$\|\nabla Q_0\|_{L^2}^2 = 1, \quad \frac{a^*}{2}\|Q_0\|_{L^4}^4 = 1.$$

Let λ_1 and λ_2 be as in (1.13). We see that

$$I_{a^*}^\delta \le \frac{\delta}{2}\lambda_1 \tau^4 + (\gamma+\epsilon)\lambda_2 \tau^{-p} + O(\tau^{-\infty})$$

as $\tau \to \infty$. We next choose $\tau = \mu \delta^{-\frac{1}{4+p}}$ for some $\mu > 0$ to be chosen shortly (note that $\tau \to \infty$ as $\delta \searrow 0$) and get

$$I_{a^*}^\delta \le \delta^{\frac{p}{4+p}} \left(\frac{\lambda_1}{2}\mu^4 + (\gamma+\epsilon)\lambda_2 \mu^{-p} \right) + O(\delta^\infty) \tag{3.9}$$

as $\delta \searrow 0$. Taking $\epsilon = \gamma$ and $\mu = 1$, we prove (3.7).

It also follows from (3.9) that

$$\delta^{-\frac{p}{4+p}} I_{a^*}^\delta \le \frac{\lambda_1}{2}\mu^4 + (\gamma+\epsilon)\lambda_2 \mu^{-p} + O(\delta^\infty)$$

which implies that

$$\limsup_{\delta \searrow 0} \delta^{-\frac{p}{4+p}} I_{a^*}^\delta \le \frac{\lambda_1}{2}\mu^4 + (\gamma+\epsilon)\lambda_2 \mu^{-p}.$$

Letting $\epsilon \to 0$ and optimizing over all $\mu > 0$, we obtain (3.8). □

Now let y_δ be as in (3.2). We write $\varepsilon_\delta y_\delta = x_\delta + z_\delta$ with $x_\delta \in [0,1]^2$ and $z_\delta \in \mathbb{Z}^2$. By the same argument as in (3.3), we prove that up to a subsequence,

$$x_\delta \to x^0 \in [0,1]^2, \quad V(x^0) = \min V = 0.$$

By (A3), we must have $x^0 \equiv x_0$.

Claim 3.4. *It holds that $\frac{x_\delta - x_0}{\varepsilon_\delta}$ is uniformly bounded as $\delta \searrow 0$. Moroever,*

$$\liminf_{\delta \searrow 0} \delta^{-\frac{p}{4+p}} I_{a^*}^\delta \ge (4+p)\left(\frac{\lambda_1}{2p}\right)^{\frac{p}{4+p}} \left(\frac{\gamma\lambda_2}{4}\right)^{\frac{4}{4+p}}. \tag{3.10}$$

Proof. If $\frac{x_\delta - x_0}{\varepsilon_\delta}$ is not bounded as $\delta \searrow 0$, then for any $M>0$, $\left|\frac{x_\delta-x_0}{\varepsilon_\delta}\right| \geq M$ as δ close to 0. It follows from the Fatou's lemma and (3.2) that

$$\begin{aligned}\lim_{\delta\searrow 0} \varepsilon_\delta^{-p} \int V(x)|\phi_\delta(x)|^2 dx &= \lim_{\delta\searrow 0} \varepsilon_\delta^{-p} \int V(\varepsilon_\delta x + y_\delta)|\psi_\delta(x+y_\delta)|^2 dx \\ &= \lim_{\delta\searrow 0} \varepsilon_\delta^{-p} \int V(\varepsilon_\delta x + x_\delta)|\psi_\delta(x+y_\delta)|^2 dx \\ &\geq \int \lim_{\delta\searrow 0} \varepsilon_\delta^{-p} V(\varepsilon_\delta x + x_\delta)|\psi_\delta(x+y_\delta)|^2 dx \\ &= \gamma \int \lim_{\delta\searrow 0} \left| x + \frac{x_\delta - x_0}{\varepsilon_\delta}\right|^p |\psi_\delta(x+y_\delta)|^2 dx \\ &\geq \gamma \int_{|x|\leq \frac{M}{2}} \lim_{\delta\searrow 0} \left| x + \frac{x_\delta - x_0}{\varepsilon_\delta}\right|^p |\psi_\delta(x+y_\delta)|^2 dx \\ &\geq \gamma \left(\frac{M}{2}\right)^p \int_{|x|\leq\frac{M}{2}} [Q_0(x)]^2 dx \geq CM^p\end{aligned}$$

for some constant $C>0$ independent of M. On the other hand,

$$\begin{aligned}\lim_{\delta\searrow 0} \varepsilon_\delta^4 \|\nabla(|\phi_\delta|^2)\|^2_{L^2} = \lim_{\delta\searrow 0} \int |\nabla(|\psi_\delta|^2)(x)|^2 dx &= \lim_{\delta\searrow 0}\int |\nabla(|\psi_\delta|^2)(x+y_\delta)|^2 dx \\ &\geq \int \lim_{\delta\searrow 0} |\nabla(|\psi_\delta|^2)(x+y_\delta)|^2 dx \\ &= \|\nabla(Q_0^2)\|^2_{L^2} = \lambda_1.\end{aligned}$$

This shows that as δ close to 0,

$$I^\delta_{a^*} = E^\delta_{a^*}(\phi_\delta) \geq A(\phi_\delta) + \frac{\delta}{2}\|\nabla(|\phi_\delta|^2)\|^2_{L^2} \geq \frac{C}{2}\varepsilon_\delta^p M^p + \frac{\delta}{4}\lambda_1 \varepsilon_\delta^{-4} \geq C(p,\lambda_1) M^{\frac{4p}{4+p}} \delta^{\frac{p}{4+p}},$$

where we have used the Young inequality in the last estimate. This however contradicts the upper bound in (3.7) for M sufficiently large. This shows that $\frac{x_\delta-x_0}{\varepsilon_\delta}$ is uniformly bounded as $\delta \searrow 0$. Passing to a subsequence if necessary, $\frac{x_\delta - x_0}{\varepsilon_\delta} \to y \in \mathbb{R}^2$ as $\delta\searrow 0$. By the same argument as above,

$$\begin{aligned}\lim_{\delta\searrow 0}\varepsilon_\delta^{-p}\int V(x)|\phi_\delta(x)|^2 dx &\geq \gamma \int \lim_{\delta\searrow 0}\left|x + \frac{x_\delta-x_0}{\varepsilon_\delta}\right|^p |\psi_\delta(x+y_\delta)|^2 dx \\ &= \gamma\int |x+y|^p [Q_0(x)]^2 dx \\ &\geq \gamma \int |x|^p [Q_0(x)]^2 dx = \gamma\lambda_2,\end{aligned} \tag{3.11}$$

where we have used the rearrangement inequality to get the last estimate. We next use the Young inequality [1] to get

$$\begin{aligned}\delta^{-\frac{p}{4+p}} I^\delta_{a^*} &\geq \delta^{-\frac{p}{4+p}} A(\phi_\delta) + \frac{\delta^{\frac{4}{4+p}}}{2}\|\nabla(|\phi_\delta|^2)\|^2_{L^2} \\ &\geq \delta^{-\frac{p}{4+p}}\left(\gamma\lambda_2 + o_{\delta\searrow 0}(1)\right)\varepsilon_\delta^p + \frac{\delta^{\frac{4}{4+p}}}{2}\left(\lambda_1 + o_{\delta\searrow 0}(1)\right)\varepsilon_\delta^{-4} \\ &\geq (4+p)\left(\frac{\delta^{\frac{4}{4+p}}}{2p}\left(\lambda_1 + o_{\delta\searrow 0}(1)\right)\varepsilon_\delta^{-4}\right)^{\frac{p}{4+p}} \left(\frac{\delta^{-\frac{p}{4+p}}}{4}\left(\gamma\lambda_2 + o_{\delta\searrow 0}(1)\right)\varepsilon_\delta^p\right)^{\frac{4}{4+p}} \\ &= (4+p)\left(\frac{\lambda_1}{2p}\right)^{\frac{p}{4+p}}\left(\frac{\gamma\lambda_2}{4}\right)^{\frac{4}{4+p}} + o_{\delta\searrow 0}(1)\end{aligned}$$

which shows (3.10). Note that the equality in the third line holds if and only if

$$\frac{\delta^{\frac{4}{4+p}}}{2}\left(\lambda_1 + o_{\delta\searrow 0}(1)\right)\varepsilon_\delta^{-4}\frac{4+p}{p} = \delta^{-\frac{p}{4+p}}\left(\gamma\lambda_2 + o_{\delta\searrow 0}(1)\right)\varepsilon_\delta^p \frac{4+p}{4}$$

[1] For any $a,b>0$ and $1<p,q<\infty$ satisfying $\frac{1}{p}+\frac{1}{q}=1$, it holds that

$$(ap)^{\frac{1}{p}}(bq)^{\frac{1}{q}} \leq a+b,$$

where the equality holds if and only if $ap = bq$.

or

$$\varepsilon_\delta = \left(\frac{2\lambda_1}{\gamma\lambda_2 p}\delta\right)^{\frac{1}{4+p}} + o_{\delta\searrow 0}(1). \tag{3.12}$$

□

By taking $\mu = \left(\frac{\gamma\lambda_2 p}{2\lambda_1}\right)^{\frac{1}{4+p}}$ in (3.8), we get

$$\limsup_{\delta\searrow 0} \delta^{-\frac{p}{4+p}} I^\delta_{a^*} \le (4+p)\left(\frac{\lambda_1}{2p}\right)^{\frac{p}{4+p}}\left(\frac{\gamma\lambda_2}{4}\right)^{\frac{4}{4+p}}$$

which together with (3.10) imply that

$$\lim_{\delta\searrow 0} \delta^{-\frac{p}{4+p}} I^\delta_{a^*} = (4+p)\left(\frac{\lambda_1}{2p}\right)^{\frac{p}{4+p}}\left(\frac{\gamma\lambda_2}{4}\right)^{\frac{4}{4+p}}$$

and (3.12) holds. We aslo have that $y \equiv 0$ in (3.11). The proof is complete. □

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