ON FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION WITH COMBINED POWER-TYPE NONLINEARITIES

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Abstract. We undertake a comprehensive study for the fractional nonlinear Schrödinger equation

$$i\partial_t u - (-\Delta)^s u = \mu_1 |u|^{\alpha_1} u + \mu_2 |u|^{\alpha_2} u, \quad u(0) = u_0,$$

where $\frac{d}{2d-1} \leq s < 1$, $0 < \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$. Firstly, we establish the local and global well-posedness results for non-radial and radial H^s initial data, radial $\dot{H}^{s_c} \cap \dot{H}^s$ initial data, where $s_c = \frac{d}{2} - \frac{2s}{\alpha_2}$. Secondly, we study the asymptotic behavior of global radial H^s solutions. Of particular interest is the L^2-critical case and the results in this case are conditional on a conjectured global existence and spacetime estimate for the L^2-critical fractional nonlinear Schrödinger equation. Thirdly, we obtain sufficient conditions about existence of blow-up radial $\dot{H}^{s_c} \cap \dot{H}^s$ solutions, and derive the sharp threshold mass of blow-up and global existence for this equation with L^2-critical and L^2-subcritical nonlinearities. Finally, we obtain the dynamical behaviour of blow-up solutions in both L^2-critical and L^2-supercritical cases, including mass-concentration and limiting profile.

1. Introduction

In recent years, there has been a great deal of interest in using fractional Laplacians to model physical phenomena. By extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, Laskin in [37, 38] used the theory of functionals over functional measure generated by the Lévy stochastic process to deduce the following nonlinear fractional Schrödinger equation

$$i\partial_t u = (-\Delta)^s u + f(u), \tag{1.1}$$

where $0 < s < 1$ and $f(u)$ is the nonlinearity. The fractional nonlinear Schrödinger equation also appears in the continuum limit of discrete models with long-range interactions (see e.g. [36]) and in the description of Bonson stars as well as in water wave dynamics (see e.g. [28]). The fractional differential operator $(-\Delta)^s$ is defined by $(-\Delta)^s u = \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}(u)]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and inverse Fourier transform, respectively.

Recently, equation (1.1) has attracted more and more attentions in both the physics and mathematics fields, see [2, 3, 7, 14, 9, 12, 15, 16, 19, 20, 21, 22, 24, 25, 26, 30, 35, 47, 32, 48, 49, 13]. For the Hartree-type nonlinearity $f(u) = \pm(|x|^{-\gamma} * |u|^2)u$, Cho et al. in [8] proved existence and uniqueness of local and global solutions of (1.1). In the focusing case, i.e. there is a minus sign in front of the nonlinearity, the existence of blow-up solutions was shown by Cho et al. in [10]. The dynamical properties of blow-up solutions have been investigated in [9, 12, 48]. Zhang and Zhu in [47] studied the stability and instability of standing waves. Guo and Zhu in [32] established the sharp threshold of blow-up and scattering in the mass-supercritical and energy-subcritical case. The global existence in the focusing energy-critical case was shown by Cho et al. in [13]. For the local nonlinearity $f(u) = \pm|u|^\alpha u$, the well-posedness and ill-posedness in the Sobolev space H^s have been investigated in [11, 34, 19]. In [3], Boulenger et al. have obtained general criteria for blow-up of radial solutions of (1.1) with the focusing nonlinearity

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$f(u)=-|u|^\alpha u$ and $\alpha \geq \frac{4s}{d}$ in $\mathbb{R}^d$, $d\geq 2$. Dynamics of blow-up solutions were studied recently by the first author in [20, 21]. The sharp threshold of blow-up and scattering in the mass-supercritical and energy-subcritical case was established in [43, 33]. Guo et al. in [30] shown the global existence and scattering in the energy-critical case. The orbital stability of standing waves for other kinds of fractional Schrödinger equations has been studied in [2, 25, 26, 49, 42].

In this paper, we study the Cauchy problem for the fractional nonlinear Schrödinger equation with combined power-type nonlinearities

$$\left\{ \begin{array}{rcl} i\partial_t u - (-\Delta)^s u &=& \mu_1 |u|^{\alpha_1} u + \mu_2 |u|^{\alpha_2} u, \quad \text{on } \mathbb{R}^+\times\mathbb{R}^d, \\ u(0) &=& u_0, \end{array}\right. \tag{1.2}$$

where $u:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{C}$, $d\geq 2$, $0<s<1$, $0<\alpha_1<\alpha_2<\frac{4s}{d-2s}$ and μ_1,μ_2 are non-zero real constants.

Throughout this paper, we define the critical Sobolev exponent associated to (1.2) by

$$s_{\rm c} := \frac{d}{2}-\frac{2s}{\alpha_2}, \tag{1.3}$$

and also the critical Lebesgue exponent associated to (1.2) by

$$\alpha_{\rm c} := \frac{2d}{d-2s_{\rm c}} = \frac{d\alpha_2}{2s}. \tag{1.4}$$

If one considers initial data in H^s, then the equation enjoys mass and energy conservation laws:

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u_0),$$
$$E(u(t)) = \frac{1}{2}\int |(-\Delta)^{s/2} u(t,x)|^2 dx + \frac{\mu_1}{\alpha_1+2}\int |u(t,x)|^{\alpha_1+2}dx + \frac{\mu_2}{\alpha_2+2}\int |u(t,x)|^{\alpha_2+2}dx = E(u_0).$$

If one considers initial data in $\dot{H}^{s_{\rm c}}\cap\dot{H}^s$, then the equation only has energy conservation. The conservation of mass is no longer available in this setting.

One of motivations for considering (1.2) is the lack of scaling invariance. It is well-known that there is a natural scaling invariance associated to the single nonlinear Schrödinger equation

$$i\partial_t u - (-\Delta)^s u = \mu|u|^\alpha u. \tag{1.5}$$

More precisely, the scaling

$$u_\lambda(t,x) := \lambda^{\frac{2s}{\alpha}} u(\lambda^{2s}t, \lambda x), \quad \lambda>0$$

leaves (1.5) invariant, that is, if u is a solution of (1.5), then u_λ is also a solution of (1.5). In our consideration with combined nonlinearities $\alpha_1<\alpha_2$, there is no scaling that leaves (1.2) invariant. However, one can use scaling and homogeneity to normalize both μ_1 and μ_2 to have magnitude one without difficulty.

When $s=1$, Tao et al. in [44] undertook a comprehensive study for the following nonlinear Schrödinger equation with combined power-type nonlinearities

$$\left\{ \begin{array}{rcl} i\partial_t u + \Delta u &=& \mu_1 |u|^{\alpha_1} u + \mu_2 |u|^{\alpha_2} u, \\ u(0) &=& u_0, \end{array}\right. \tag{1.6}$$

where $0<\alpha_1<\alpha_2\leq\frac{4}{d-2}, d\geq 3$ and μ_1,μ_2 are non-zero real numbers. More precisely, they addressed questions related to local and global well-posedness, finite time blow-up in weighted space $\Sigma := H^1\cap L^2(|x|^2dx)$, and asymptotic behaviour (scattering) in both energy space H^1 and weighted space Σ. The scattering versus blow-up for some particular cases of (1.6) was studied in [40, 46, 6, 41]. Recently, in [23], the second author proved the existence of blow-up solutions and found the sharp threshold of blow-up and global existence for (1.6) with $0<\alpha_1<\frac{4}{d}$, $\alpha_2=\frac{4}{d}$, $\mu_1>0$ and $\mu_2<0$, which is a complement to the result in [44].

For the fractional nonlinear Schrödinger equation (1.2), the second author in [24] established some sufficient conditions about the existence of blow-up solutions, sharp thresholds of blow-up and global existence and dynamical properties of blow-up solutions in the L^2-critical case, i.e. $0 < \alpha_1 < \frac{4s}{d}$, $\alpha_2 = \frac{4s}{d}$, $\mu_1 > 0$ and $\mu_2 < 0$, including mass-concentration, blow-up rates, and limiting profile.

In this paper, we will systematically study the Cauchy problem (1.2). We are interested in local and global well-posedness, asymptotic behavior(scattering), the existence of finite time blow-up solutions and dynamical properties of blow-up solutions in the L^2-critical and L^2-supercritical cases, including mass-concentration and limiting profile. We also mention that in this paper, we do not consider the energy-critical case, i.e. $\alpha_2 = \frac{4s}{d-2s}$. The reasons for that are the lack of dispersive estimates for the fractional Schrödinger operator $e^{-it(-\Delta)^s}$ as well as the lack of a good global theory for the single energy-critical equation. We hope to consider this interesting problem in a future work.

Firstly, applying a fixed point argument and Strichartz estimates, we establish the local well-posedness results of (1.2) for non-radial and radial H^s initial data, radial $\dot{H}^{s_c} \cap \dot{H}^s$ initial data. These results are complements to the ones in [24, 49]. Note that non-radial Strichartz estimates for the fractional Schrödinger equation are well-known to have a loss of derivatives. This is the main reason why we mainly consider radial data in this paper. We refer the reader to Section 3 for more details.

Secondly, using some elementary inequalities, we establish an a priori estimate on the kinetic energy, namely

$$\sup_{t\in[0,T)} \|u(t)\|_{\dot{H}^s} \leq C(E, M),$$

where E and M are the conserved energy and mass respectively. With this a priori bound, the blow-up alternative yields the global existence of H^s solutions to (1.2) in two cases:

- $0 < \alpha_1 < \alpha_2 < \frac{4s}{d}$ and $\mu_1, \mu_2 \in \mathbb{R}$;
- $0 < \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$, $\mu_1 \in \mathbb{R}$ and $\mu_2 > 0$.

We refer the reader to Section 6 for more details. We also give some criteria for the global existence of radial $\dot{H}^{s_c} \cap \dot{H}^s$ solutions to (1.2) at the end of Section 8.

Thirdly, we will study the asymptotic behavior of global radial H^s solutions to (1.2) with $\frac{4s}{d} \leq \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$ and $\mu_2 > 0$. In the case $\alpha_1 = \frac{4s}{d}$, the L^2-stability is exploited. To do so, by a similar idea as in [44], we need to assume a good global theory for the single defocusing L^2-critical FNLS, namely

$$\begin{cases} i\partial_t v - (-\Delta)^s v &= |v|^{\frac{4s}{d}} v, \quad \text{on } \mathbb{R}^+ \times \mathbb{R}^d, \\ v(0) &= v_0. \end{cases} \tag{1.7}$$

More precisely, we need the following assumption.

Assumption 1.1. Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$ and $v_0 \in H^s$ be radial. Then there exists a unique global solution v to (1.7) and the global solution satisfies

$$\|v\|_{L^{\frac{2(d+2s)}{d}}(\mathbb{R}^+\times\mathbb{R}^d)} \leq C(\|v_0\|_{L^2}).$$

Unlike the nonlinear Schrödinger equation $s = 1$, equation (1.2) does not enjoy the a priori interaction Morawetz estimate. However, thanks to the radial assumption, we are able to derive a priori radial Morawetz estimates in the defocusing case, i.e. $\mu_1, \mu_2 > 0$ and $\frac{4s}{d} \leq \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$. This estimate allows us to control the solution in the L^m-norm with $\alpha_1 + 1 + \frac{2}{d-2s} \leq m \leq \alpha_2 + 1 + \frac{2}{d-2s}$. Using this global L^m-norm bound, we can derive the global Strichartz bound of solutions. With the help of this global Strichartz bound, the scattering follows easily. We refer the reader to Section 7 for more details.

Fourthly, we will investigate sufficient conditions about the existence of blow-up radial $\dot{H}^{s_c} \cap \dot{H}^s$ solutions for (1.2) by using the method of Boulenger et al. [3]. When $\mu_1 > 0$, $0 < \alpha_1 < \alpha_2 = \frac{4s}{d}$ and

$\mu_2 < 0$, Feng in [24] found the sharp threshold mass $\|Q\|_{L^2}$ of blow-up and global existence for (1.2), where Q is the ground state solution of (2.11) with $\alpha = \frac{4s}{d}$. However, there is an error in the proof of the existence of blow-up solutions given in [24]. In addition, in this case, it follows from Lemma 4.4 that

$$\frac{d}{dt}M_{\varphi_R}(u(t)) \leq 8sE(u(t)) + \mu_1\frac{2(d\alpha_1 - 4s)}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} \leq 8sE(u_0).$$

Therefore, by using the argument of Boulenger et al [3], we can prove the existence of blow-up solutions by choosing the initial data u_0 such that $E(u_0) < 0$. But when $\mu_1 < 0$ and $0 < \alpha_1 < \frac{4s}{d}$, we have

$$\frac{d}{dt}M_{\varphi_R}(u(t)) \leq 8sE(u(t)) + \mu_1\frac{2(d\alpha_1 - 4s)}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}}.$$

Because $\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}}$ is a positive uncertain function, which may be bounded or not relative to t. Hence, it is hard to choose $E(u_0)$ to ensure the existence of blow-up solutions. We develop a new argument by contradiction to solve this problem. In addition, our method can be easily applied to prove the existence of blow-up solutions for (1.6) with $\mu_1 < 0$, $\mu_2 < 0$ and $0 < \alpha_1 < \alpha_2 = \frac{4}{d}$, which is an open problem left by Tao et al. in [44]. As far as we know, this result has not been proved yet. Therefore, this type of result for (1.2) is new even if $s = 1$.

Finally, we obtain the dynamical behaviour of blow-up solutions to (1.2) in both L^2-critical and L^2-supercritical cases, including mass-concentration and limiting profile. In the L^2-critical case, i.e. $0 < \alpha_1 < \alpha_2 = \frac{4s}{d}$, the second author in [24] studied dynamical properties of finite time blow-up solutions with $\mu_1 > 0$ and $\mu_2 < 0$. In this paper, we extend the result of [24] to $\mu_1 \in \mathbb{R}$. In the L^2-supercritical case, i.e. $\frac{4s}{d} < \alpha_2 < \frac{4s}{d-2s}$ and $0 < \alpha_1 < \alpha_2$, since the uniqueness of solutions to elliptic equations (2.14) and (2.17) are not known yet, we need to introduce notions of Sobolev and Lebesgue ground states in order to describe the dynamical behaviour of the blow-up solutions to (1.2) in the homogenoues setting.

This paper is organized as follows. In Section 2, we present some preliminaries including Strichartz estimates, profile decompositions and compactness lemmas. In Section 3, we establish the local well-posedness results of (1.2) for non-radial and radial H^s initial data as well as radial $\dot{H}^{s_c} \cap \dot{H}^s$ initial data. In Section 4, we recall and establish some new virial estimates related to (1.2) for both radial H^s data and radial $\dot{H}^{s_c} \cap \dot{H}^s$ data. In Section 5, we study the stability of the L^2-critical fractional nonlinear Schrödinger equation. In Section 6, we establish the global well-posedness of radial H^s solutions to (1.2). In Section 7, we show the asymptotic behavior in the energy space for global radial H^s solutions to (1.2). In Section 8, we will establish some sufficient conditions about the existence of blow-up solutions for (1.2), and then obtain some sharp thresholds of blow-up and global existence. In Section 9, we study the dynamical behaviour of the blow-up solutions to (1.2) in both L^2-critical and L^2-supercritical cases, including mass-concentration and limiting profile.

2. Preliminaries

2.1. **Sobolev spaces.** We first recall the definition of generalized inhomogeous and homogeneous Sobolev spaces used in this paper (see e.g. [1, Chapter 6], [29, Appendix] and [45, Chapter 5]). Let $\mathcal{S}$ be the space of Schwartz functions. For $\gamma \in \mathbb{R}$ and $1 \leq q \leq \infty$, the generalized Sobolev space $W^{\gamma,q}$ is defined as a closure of $\mathcal{S}$ under the norm

$$\|u\|_{W^{\gamma,q}} := \|\langle\nabla\rangle^\gamma u\|_{L^q},$$

where $|\nabla| = \sqrt{-\Delta}$ and $\langle x\rangle = \sqrt{1+|x|^2}$ is the Japanese bracket. Denote $\mathcal{S}_0$ the subspace of $\mathcal{S}$ consisting of functions ϕ satisfying $D^\alpha\hat{\phi}(0) = 0$ for all $\alpha \in \mathbb{N}^d$, where $\hat{\cdot}$ is the Fourier transform on $\mathcal{S}$. We define the generalized homogeneous Sobolev space $\dot{W}^{\gamma,q}$ as a closure of $\mathcal{S}_0$ under the norm

$$\|u\|_{\dot{W}^{\gamma,q}} := \||\nabla|^\gamma u\|_{L^q}.$$

Under the above setting, the spaces $W^{\gamma,q}$ and $\dot{W}^{\gamma,q}$ are Banach spaces. Moreover, for $\gamma \geq 0$, we have $W^{\gamma,q} = L^q \cap \dot{W}^{\gamma,q}$. We shall use $H^\gamma = W^{\gamma,2}$ and $\dot{H}^\gamma = \dot{W}^{\gamma,2}$. Note that if $\gamma_1 \leq \gamma_2$, then $H^{\gamma_2} \subset H^{\gamma_1}$. However, the spaces $\dot{H}^{\gamma_1}$ and $\dot{H}^{\gamma_2}$ cannot be compared for the inclusion. Nevertheless, for $\gamma_1 < \gamma < \gamma_2$, the space $\dot{H}^\gamma$ is an interpolation space between $\dot{H}^{\gamma_1}$ and $\dot{H}^{\gamma_2}$.

2.2. **Strichartz estimates.** For $J \subset \mathbb{R}$ and $p, q \in [1, \infty]$, we define the mixed norm

$$\|u\|_{L^p(J,L^q)} := \left(\int_J \left(\int_{\mathbb{R}^d} |u(t,x)|^q dx \right)^{\frac{p}{q}} \right)^{\frac{1}{p}},$$

with the usual modification when either p or q are infinity. In the case $p = q$, we shall use $L^p(J \times \mathbb{R}^d)$ instead of $L^p(J, L^p)$. We next recall Strichartz estimates for the unitary group $e^{-it(-\Delta)^s}$. It is well-known that $e^{-it(-\Delta)^s}$ enjoys several types of Strichartz estimates, in particular non-radial and radial Strichartz estimates which are recalled as follows.

- **Non-radial Strichartz estimates** [16, 19]**:** for $d \geq 1$, $s \in (0,1)\backslash\{1/2\}$,

$$\|e^{-it(-\Delta)^s} u_0\|_{L^p(\mathbb{R},L^q)} \lesssim \||\nabla|^{s_{p,q}} u_0\|_{L^2}, \tag{2.1}$$

$$\left\| \int_0^t e^{-i(t-\tau)(-\Delta)^s} f(\tau) d\tau \right\|_{L^p(\mathbb{R},L^q)} \lesssim \||\nabla|^{s_{p,q} - s_{a',b'} - 2s} f\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{2.2}$$

  where (p,q) and (a,b) are Schrödinger admissible pairs, i.e.

$$p \in [2,\infty], \quad q \in [2,\infty), \quad (p,q,d) \neq (2,\infty,2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}, \tag{2.3}$$

  and

$$s_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{2s}{p}, \quad s_{a',b'} = \frac{d}{2} - \frac{d}{b'} - \frac{2s}{a'}. \tag{2.4}$$

  Since $\frac{d}{2} - \frac{d}{q} - \frac{2}{p} + \frac{2-2s}{p} \geq \frac{2-2s}{p}$, we see that $s_{p,q} > 0$ for any Schrödinger admissible pairs except $(p,q) = (\infty, 2)$. This shows that non-radial Strichartz estimates have a loss of derivatives. This loss of derivatives makes the study of local well-posedness with non-radial data more difficult. We refer to Section 3 for more details.
- **Radial Strichartz estimates** [31, 35, 14]**:** for $d \geq 2$, $s \in (0,1)\backslash\{1/2\}$, the estimates (2.1) and (2.2) hold true provided (p,q) and (a,b) satisfy the radial Schrödinger admssible condition

$$p \in [2,\infty], \quad q \in [2,\infty), \quad (p,q) \neq \left(2, \frac{4d-2}{2d-3}\right), \quad \frac{2}{p} + \frac{2d-1}{q} \leq \frac{2d-1}{2}. \tag{2.5}$$

  The last condition in (2.5) allows us to choose (p,q) so that $s_{p,q} = 0$. Pluging it into $\frac{2}{p} + \frac{2d-1}{q} \leq \frac{2d-1}{2}$, we have the following radial Strichartz estimates: for $d \geq 2$ and $\frac{d}{2d-1} \leq s < 1$,

$$\|e^{-it(-\Delta)^s} u_0\|_{L^p(\mathbb{R},L^q)} \lesssim \|u_0\|_{L^2}, \tag{2.6}$$

$$\left\| \int_0^t e^{-i(t-\tau)(-\Delta)^s} f(\tau) d\tau \right\|_{L^p(\mathbb{R},L^q)} \lesssim \|f\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{2.7}$$

  where u_0 and f are radially symmetric and $(p,q), (a,b)$ are fractional admissible

$$p \in [2,\infty], \quad q \in [2,\infty), \quad (p,q) \neq \left(2, \frac{4d-2}{2d-3}\right), \quad \frac{2s}{p} + \frac{d}{q} = \frac{d}{2}. \tag{2.8}$$

  These Strichartz estimates without loss of derivatives allow us to give a better local well-posedness result. This is the reason why we mainly consider radially symmetric initial data throughout this paper.

2.3. Profile decompositions.

Lemma 2.1 (H^s profile decomposition [48, 20]). *Let $d \geq 2$ and $0 < s < 1$. Let $(v_n)_{n\geq 1}$ be a bounded sequence in H^s. Then there exist a subsequence still denoted by $(v_n)_{n\geq 1}$, a family $(x_n^j)_{j\geq 1}$ of sequences in $\mathbb{R}^d$ and a sequence $(V^j)_{j\geq 1}$ of H^s functions such that*

- *for every $k \neq j$,*
$$|x_n^k - x_n^j| \to \infty, \quad \text{as } n \to \infty;$$
- *for every $l \geq 1$ and every $x \in \mathbb{R}^d$,*
$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x),$$
with
$$\limsup_{n\to\infty} \|v_n^l\|_{L^q} \to 0, \quad \text{as } l \to \infty,$$
for every $2 < q < \frac{2d}{d-2s}$.

Moreover, for every $l \geq 1$,
$$\|v_n\|_{L^2}^2 = \sum_{j=1}^{l} \|V^j\|_{L^2}^2 + \|v_n^l\|_{L^2}^2 + o_n(1),$$
$$\|v_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{l} \|V^j\|_{\dot{H}^s}^2 + \|v_n^l\|_{\dot{H}^s}^2 + o_n(1),$$
as $n \to \infty$.

Lemma 2.2 ($\dot{H}^{\gamma_c} \cap \dot{H}^s$ profile decomposition [21]). *Let $d \geq 2$, $0 < s < 1$, $\frac{4s}{d} < \alpha < \frac{4s}{d-2s}$. Denote*
$$\gamma_c := \frac{d}{2} - \frac{2s}{\alpha}, \quad \beta_c := \frac{2d}{d-2\gamma_c} = \frac{d\alpha}{2s}. \tag{2.9}$$
Let $(v_n)_{n\geq 1}$ be a bounded sequence in $\dot{H}^{\gamma_c} \cap \dot{H}^s$. Then there exist a subsequence still denoted by $(v_n)_{n\geq 1}$, a family $(x_n^j)_{j\geq 1}$ of sequences in $\mathbb{R}^d$ and a sequence $(V^j)_{j\geq 1}$ of $\dot{H}^{\gamma_c} \cap \dot{H}^s$ functions such that

- *for every $k \neq j$,*
$$|x_n^k - x_n^j| \to \infty, \quad \text{as } n \to \infty;$$
- *for every $l \geq 1$ and every $x \in \mathbb{R}^d$,*
$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x),$$
with
$$\limsup_{n\to\infty} \|v_n^l\|_{L^q} \to 0, \quad \text{as } l \to \infty,$$
for every $\beta_c < q < \frac{2d}{d-2s}$.

Moreover, for every $l \geq 1$,
$$\|v_n\|_{\dot{H}^{\gamma_c}}^2 = \sum_{j=1}^{l} \|V^j\|_{\dot{H}^{\gamma_c}}^2 + \|v_n^l\|_{\dot{H}^{\gamma_c}}^2 + o_n(1),$$
$$\|v_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{l} \|V^j\|_{\dot{H}^s}^2 + \|v_n^l\|_{\dot{H}^s}^2 + o_n(1),$$
as $n \to \infty$.

2.4. **Sharp Gagliardo-Nirenberg inequality.** A first application of the profile decomposition is the following sharp Gagliardo-Nirenberg inequalities.

Lemma 2.3 ([3, 49])**.** *Let $d \geq 2$, $0 < s < 1$ and $0 < \alpha < \frac{4s}{d-2s}$. Then for any $u \in H^s$,*

$$\|u\|^{\alpha+2}_{L^{\alpha+2}} \leq C_{\rm opt} \|u\|^{\frac{d\alpha}{2s}}_{\dot{H}^s} \|u\|^{\alpha+2-\frac{d\alpha}{2s}}_{L^2}, \tag{2.10}$$

where the optimal constant $C_{\rm opt}$ is given by

$$C_{\rm opt} = \left(\frac{2s(\alpha+2)-d\alpha}{d\alpha}\right)^{\frac{d\alpha}{4s}} \frac{2s(\alpha+2)}{2s(\alpha+2)-d\alpha} \frac{1}{\|Q\|^{\alpha}_{L^2}}.$$

Here Q is the unique (up to symmetries) positive radial solution to the elliptic equation

$$(-\Delta)^s Q + Q - |Q|^\alpha Q = 0. \tag{2.11}$$

Moreover, the following Pohozaev's identities hold true:

$$\|Q\|^2_{\dot{H}^s} = \frac{d\alpha}{2s(\alpha+2)} \|Q\|^{\alpha+2}_{L^{\alpha+2}} = \frac{d\alpha}{4s-(d-2s)\alpha} \|Q\|^2_{L^2}. \tag{2.12}$$

Remark 2.4. The uniqueness of positive radial solution to (2.11) was shown recently in [22, 27]. Note that the estimate (2.10) still holds true in one dimension (see e.g. [22]).

Lemma 2.5 ([21])**.** *Let $d \geq 2$, $0 < s < 1$ and $\frac{4s}{d} < \alpha < \frac{4s}{d-2s}$ and $\gamma_{\rm c}, \beta_{\rm c}$ be as in (2.9).*

- *Then for any $u \in \dot{H}^{\gamma_{\rm c}} \cap \dot{H}^s$,*

$$\|u\|^{\alpha+2}_{L^{\alpha+2}} \leq A_{\rm opt} \|u\|^{\alpha}_{\dot{H}^{\gamma_{\rm c}}} \|u\|^2_{\dot{H}^s}, \tag{2.13}$$

where the optimal constant $A_{\rm opt}$ is given by

$$A_{\rm opt} = \frac{\alpha+2}{2} \frac{1}{\|W\|^{\alpha}_{\dot{H}^{\gamma_{\rm c}}}},$$

with W a solution to the elliptic equation

$$(-\Delta)^s W + (-\Delta)^{\gamma_{\rm c}} W - |W|^\alpha W = 0. \tag{2.14}$$

Moreover, the following Pohozaev's identities hold true:

$$\|W\|^2_{\dot{H}^s} = \frac{2}{\alpha+2} \|W\|^{\alpha+2}_{L^{\alpha+2}} = \frac{2}{\alpha} \|W\|^2_{\dot{H}^{\gamma_{\rm c}}}. \tag{2.15}$$

- *Then for any $u \in L^{\beta_{\rm c}} \cap \dot{H}^s$,*

$$\|u\|^{\alpha+2}_{L^{\alpha+2}} \leq B_{\rm opt} \|u\|^{\alpha}_{L^{\beta_{\rm c}}} \|u\|^2_{\dot{H}^s}, \tag{2.16}$$

where the optimal constant $B_{\rm opt}$ is given by

$$B_{\rm opt} = \frac{\alpha+2}{2} \frac{1}{\|R\|^{\alpha}_{L^{\beta_{\rm c}}}},$$

with R a solution to the elliptic equation

$$(-\Delta)^s R + |R|^{\beta_{\rm c}} R - |R|^\alpha R = 0. \tag{2.17}$$

Moreover, the following Pohozaev's identities hold true:

$$\|R\|^2_{\dot{H}^s} = \frac{2}{\alpha+2} \|R\|^{\alpha+2}_{L^{\alpha+2}} = \frac{2}{\alpha} \|R\|^2_{L^{\beta_{\rm c}}}. \tag{2.18}$$

Since the uniqueness of solutions to (2.14) and (2.17) are still unknown. To study dynamical properties of blow-up solutions in the homogeneous setting, we need to introduce the notions of ground states. Denote

$$G(u) := \|u\|^{\alpha+2}_{L^{\alpha+2}} \div \left[\|u\|^{\alpha}_{\dot{H}^{\gamma_c}} \|u\|^2_{\dot{H}^s}\right], \quad u \in \dot{H}^{\gamma_c} \cap \dot{H}^s,$$
$$K(u) := \|u\|^{\alpha+2}_{L^{\alpha+2}} \div \left[\|u\|^{\alpha}_{L^{\beta_c}} \|u\|^2_{\dot{H}^s}\right], \quad u \in L^{\beta_c} \cap \dot{H}^s.$$

Definition 2.6 (Ground states). (1) *We call* **Sobolev ground states** *the maximizers of G which are solutions to* (2.14)*. We denote the set of Sobolev ground states by $\mathcal{G}$.*

(2) *We call* **Lebesgue ground states** *the maximizers of K which are solutions to* (2.17)*. We denote the set of Lebesgue ground states by $\mathcal{K}$.*

It follows from the definition of ground states that if g and k are Sobolev ground state and Lebesgue ground state respectively, then

$$A_{\text{opt}} = \frac{\alpha+2}{2}\|g\|^{-\alpha}_{\dot{H}^{\gamma_c}}, \quad B_{\text{opt}} = \frac{\alpha+2}{2}\|k\|^{-\alpha}_{L^{\beta_c}}.$$

This implies that all Sobolev ground states have the same $\dot{H}^{\gamma_c}$-norm and all Lebesgue ground states have the same L^{β_c}-norm. We thus denote

$$S_{\text{gs}} := \|g\|_{\dot{H}^{\gamma_c}}, \quad \forall g \in \mathcal{G}, \tag{2.19}$$
$$L_{\text{gs}} := \|k\|_{L^{\beta_c}}, \quad \forall k \in \mathcal{K}. \tag{2.20}$$

The sharp Gagliardo-Nirenberg inequalities (2.13) and (2.16) can be written as

$$\|u\|^{\alpha+2}_{L^{\alpha+2}} \leq \frac{\alpha+2}{2}\left(\frac{\|u\|_{\dot{H}^{\gamma_c}}}{S_{\text{gs}}}\right)^{\alpha}\|u\|^2_{\dot{H}^s}, \tag{2.21}$$
$$\|u\|^{\alpha+2}_{L^{\alpha+2}} \leq \frac{\alpha+2}{2}\left(\frac{\|u\|_{L^{\beta_c}}}{L_{\text{gs}}}\right)^{\alpha}\|u\|^2_{\dot{H}^s}. \tag{2.22}$$

2.5. **Compactness lemmas.** Another application of the profile decomposition is the following compactness lemmas.

Lemma 2.7 (Compactness lemma I [24, 20]). *Let $d \geq 2$ and $0 < s < 1$. Let $(v_n)_{n\geq 1}$ be a bounded sequence in H^s such that*

$$\limsup_{n\to\infty}\|v_n\|_{\dot{H}^s} \leq M, \quad \limsup_{n\to\infty}\|v_n\|_{L^{\frac{4s}{d}+2}} \geq m.$$

Then there exists a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^d$ such that up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } H^s,$$

for some $V \in H^s$ satisfying

$$\|V\|^{\frac{4s}{d}}_{L^2} \geq \frac{d}{d+2s}\frac{m^{\frac{4s}{d}+2}}{M^2}\|Q\|^{\frac{4s}{d}}_{L^2}, \tag{2.23}$$

where Q is the unique (up to symmetries) positive radial solution to the elliptic equation (2.11) *with $\alpha = \frac{4s}{d}$.*

Remark 2.8. The lower bound (2.23) is optimal. Indeed, taking $v_n = Q$ where Q is given in 2.3, we get the equality.

Lemma 2.9 (Compactness lemma II [21]). *Let $d \geq 2$ and $0 < s < 1$, $\frac{4s}{d} < \alpha < \frac{4s}{d-2s}$ and γ_c, β_c be as in* (2.9)*. Let $(v_n)_{n\geq 1}$ be a bounded sequence in $\dot{H}^{\gamma_c} \cap \dot{H}^s$ such that*

$$\limsup_{n\to\infty}\|v_n\|_{\dot{H}^s} \leq M, \quad \limsup_{n\to\infty}\|v_n\|_{L^{\alpha+2}} \geq m.$$

- *Then there exists a sequence* $(y_n)_{n\geq 1}$ *in* $\mathbb{R}^d$ *such that up to a subsequence,*
$$v_n(\cdot + y_n) \rightharpoonup P \text{ weakly in } \dot{H}^{\gamma_c} \cap \dot{H}^s,$$
for some $P \in \dot{H}^{\gamma_c} \cap \dot{H}^s$ *satisfying*
$$\|P\|^{\alpha}_{\dot{H}^{\gamma_c}} \geq \frac{2}{\alpha+2}\frac{m^{\alpha+2}}{M^2} S^{\alpha}_{\mathrm{gs}}. \tag{2.24}$$
- *Then there exists a sequence* $(z_n)_{n\geq 1}$ *in* $\mathbb{R}^d$ *such that up to a subsequence,*
$$v_n(\cdot + z_n) \rightharpoonup Q \text{ weakly in } L^{\beta_c} \cap \dot{H}^s,$$
for some $Q \in L^{\beta_c} \cap \dot{H}^s$ *satisfying*
$$\|Q\|^{\alpha}_{L^{\beta_c}} \geq \frac{2}{\alpha+2}\frac{m^{\alpha+2}}{M^2} L^{\alpha}_{\mathrm{gs}}. \tag{2.25}$$

Remark 2.10. The lower bounds (2.24) and (2.25) are optimal. Indeed, taking $v_n = W$ in the first case and $v_n = R$ in the second case where W, R are given in Lemma 2.5, we get the equalities.

3. Local well-posedness

In this section, we establish the local well-posedness for (1.2) in the case of non-radial, radial H^s initial data as well as radial $\dot{H}^{s_c} \cap \dot{H}^s$ initial data. The proofs are based on Strichartz estimates and the standard fixed point argument, so we will omit them and only give some comments.

3.1. Non-radial H^s initial data.

Proposition 3.1 (Non-radial H^s LWP)**.** *Let* $s \in (0,1)\backslash\{1/2\}$ *and* $0 < \alpha_1 < \alpha_2$ *be such that*
$$s > \begin{cases} \frac{1}{2} - \frac{2s}{\max(\alpha_2,4)} & \text{if } d = 1, \\ \frac{d}{2} - \frac{2s}{\max(\alpha_2,2)} & \text{if } d \geq 2. \end{cases} \tag{3.1}$$
Then for all $u_0 \in H^s$*, there exist* $T \in (0,+\infty]$ *and a unique solution to* (1.2) *satisfying*
$$u \in C([0,T), H^s) \cap L^p_{\mathrm{loc}}([0,T), L^\infty),$$
for some $p > \max(\alpha_2, 4)$ *when* $d = 1$ *and some* $p > \max(\alpha_2, 2)$ *when* $d \geq 2$*. Moreover, the following properties hold:*

- *If* $T < +\infty$*, then* $\|u(t)\|_{H^s} \to \infty$ *as* $t \uparrow T$*.*
- *The solution enjoys conservation of mass and energy, i.e.* $M(u(t)) = M(u_0)$ *and* $E(u(t)) = E(u_0)$ *for all* $t \in [0,T)$*.*

We refer the reader to [20] (or [19]) for the proof of this result. Note that in the case of non-radial H^s initial data, Strichartz estimates have a loss of derivatives. However, the loss of derivatives can be compensated by using the Sobolev embedding.

Remark 3.2. It follows from (3.1) and $s \in (0,1)\backslash\{1/2\}$ that the local well-posedness for non-radial H^s initial data is available only for
$$\begin{cases} 1/3 < s < 1/2, & 0 < \alpha_1 < \alpha_2 < \frac{4s}{1-2s} & \text{if } d = 1, \\ 1/2 < s < 1, & 0 < \alpha_1 < \alpha_2 < \infty & \text{if } d = 1, \\ d/4 < s < 1, & 0 < \alpha_1 < \alpha_2 < \frac{4s}{d-2s} & \text{if } d = 2,3. \end{cases} \tag{3.2}$$

3.2. Radial H^s initial data.

Proposition 3.3 (Radial H^s LWP). *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$ and $0 < \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$. Let*

$$p_j = \frac{4s(\alpha_j + 2)}{\alpha_j(d-2s)}, \quad q_j = \frac{d(\alpha_j+2)}{d+\alpha_j s}, \quad j = 1, 2. \tag{3.3}$$

Then for any $u_0 \in H^s$ radial, there exist $T \in (0, +\infty]$ and a unique solution to (1.2) satisfying

$$u \in C([0,T), H^s) \cap_{j=1,2} L^{p_j}([0,T), W^{s,q_j}).$$

Moreover, the following properties hold:

- *$u \in L^a_{\text{loc}}([0,T), W^{s,b})$ for any fractional admissible pair (a,b).*
- *If $T < +\infty$, then $\|u(t)\|_{\dot{H}^s} \to \infty$ as $t \uparrow T$.*
- *The solution enjoys conservation of mass and energy, i.e. $M(u(t)) = M(u_0)$ and $E(u(t)) = E(u_0)$ for all $t \in [0,T)$.*

We again refer the reader to [20] for the proof of this result. In this case, Strichartz estimates have no loss of derivatives. We thus get a better local well-posedness result compared to the one in Proposition 3.1.

3.3. Radial $\dot{H}^{s_c} \cap \dot{H}^s$ initial data.

Proposition 3.4 (Radial $\dot{H}^{s_c} \cap \dot{H}^s$ LWP). *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $\frac{4s}{d} < \alpha_2 < \frac{4s}{d-2s}$, $\frac{d\alpha_2 - 4s}{2s} \leq \alpha_1 < \alpha_2$ and s_c be as in (1.3). Let (p_j, q_j) be as in (3.3). Then for any $u_0 \in \dot{H}^{s_c} \cap \dot{H}^s$ radial, there exist $T > 0$ and a unique solution u to (1.2) satisfying*

$$u \in C([0,T), \dot{H}^{s_c} \cap \dot{H}^s) \cap \cap_{j=1,2} L^{p_j}_{\text{loc}}([0,T), \dot{W}^{s_c,q_j} \cap \dot{W}^{s,q_j}).$$

Moreover, the following properties hold:

- *$u \in L^a_{\text{loc}}([0,T), \dot{W}^{s_c,b} \cap \dot{W}^{s,b})$ for any fractional admissible pair (a,b).*
- *The solution enjoys the conservation of energy, i.e. $E(u(t)) = E(u_0)$ for all $t \in [0,T)$.*
- *If $T < \infty$, then $\|u(t)\|_{\dot{H}^{s_c}} + \|u(t)\|_{\dot{H}^s} \to \infty$ as $t \uparrow T$.*

Proof. The proof is similar to the one of Proposition 3.3 (see also [21]). We thus omit the details. Note that the condition

$$\frac{d\alpha_2 - 4s}{2s} \leq \alpha_1 < \alpha_2$$

ensures that $u \in C([0,T), L^{\alpha_1+2} \cap L^{\alpha_2+2})$, hence the energy functional is well-defined. Note also that the conservation of mass is no longer available in this setting. □

4. Virial estimates

In this section, we recall virial estimates related to (1.2). Let us start with the following estimate.

Lemma 4.1 ([3]). *Let $d \geq 1$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ be such that $\nabla\varphi \in W^{1,\infty}$. Then for all $u \in H^{1/2}$,*

$$\left| \int \overline{u}(x) \nabla\varphi(x) \cdot \nabla u(x) dx \right| \leq C \left(\||\nabla|^{1/2} u\|^2_{L^2} + \|u\|_{L^2} \||\nabla|^{1/2} u\|_{L^2} \right),$$

for some $C > 0$ depending only on $\|\nabla\varphi\|_{W^{1,\infty}}$ and d.

4.1. H^s **virial estimates.** Let $d \geq 1$, $1/2 \leq s < 1$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ be such that $\nabla\varphi \in W^{3,\infty}$. Assume [1] $u \in C([0,T), H^s)$ is a solution to (1.2). The localized virial action of u is defined by

$$M_\varphi(u(t)) := 2\int \nabla\varphi(x) \cdot \operatorname{Im}\left(\overline{u}(t,x)\nabla u(t,x)\right)dx. \tag{4.1}$$

It follows from Lemma 4.1 that $M_\varphi(u(t))$ is well-defined. Indeed, by Lemma 4.1,

$$|M_\varphi(u(t))| \lesssim C\left(\|\nabla\varphi\|_{L^\infty}, \|\Delta\varphi\|_{L^\infty}\right)\|u(t)\|^2_{H^{1/2}} \lesssim C(\varphi)\|u(t)\|^2_{H^s} < \infty.$$

To study the time evolution of $M_\varphi(u(t))$, we need the following auxiliary function

$$u_m(t,x) := c_s \frac{1}{-\Delta + m}u(t,x) = c_s\mathcal{F}^{-1}\left(\frac{\hat{u}(t,\xi)}{|\xi|^2+m}\right), \quad m > 0, \tag{4.2}$$

where

$$c_s := \sqrt{\frac{\sin \pi s}{\pi}}.$$

Remark that since $u(t) \in H^s$, the smoothing property of $(-\Delta+m)^{-1}$ implies that $u_m(t) \in H^{s+2}$ for any $t \in [0,T)$.

Lemma 4.2 ([3]). *For any $t \in [0,T)$, the following identity holds true*

$$\begin{aligned}\frac{d}{dt}M_\varphi(u(t)) = &-\int_0^\infty m^s \int \Delta^2\varphi|u_m(t)|^2dxdm + 4\sum_{j,k=1}^d \int_0^\infty m^s\int \partial^2_{jk}\varphi\partial_j\overline{u}_m(t)\partial_k u_m(t)dxdm\\ &+\frac{2\mu_1\alpha_1}{\alpha_1+2}\int \Delta\varphi|u(t)|^{\alpha_1+2}dx + \frac{2\mu_2\alpha_2}{\alpha_2+2}\int\Delta\varphi|u(t)|^{\alpha_2+2}dx,\end{aligned} \tag{4.3}$$

where u_m is defined in (4.2).

Using Plancherel's and Fubini's theorem, it follows that

$$\begin{aligned}\int_0^\infty m^s\int|\nabla u_m|^2dxdm &= \int\left(\frac{\sin\pi s}{\pi}\int_0^\infty\frac{m^s dm}{(|\xi|^2+m)^2}\right)|\xi|^2|\hat{u}(\xi)|^2d\xi\\ &= \int(s|\xi|^{2s-2})|\xi|^2|\hat{u}(\xi)|^2d\xi = s\|u\|^2_{\dot{H}^s}.\end{aligned} \tag{4.4}$$

If we make formal substitution and take the unbouned function $\nabla\varphi(x) = 2x$, then we have $\partial^2_{jk}\varphi = 2\delta_{jk}$ and $\Delta^2\varphi = 0$. Using (4.4), we find formally the virial identity

$$\begin{aligned}\frac{d}{dt}M_{|x|^2}(u(t)) &= 8s\|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} + \frac{4d\mu_2\alpha_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}}\\ &= 4d\alpha_2E(u(t)) - 2(d\alpha_2-4s)\|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1(\alpha_1-\alpha_2)}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}}.\end{aligned} \tag{4.5}$$

Now let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be as above. We assume in addition that φ is radially symmetric and satisfies

$$\varphi(r) := \begin{cases} r^2 & \text{for } r \leq 1,\\ \text{const.} & \text{for } r \geq 10,\end{cases} \quad \text{and } \varphi''(r) \leq 2 \text{ for } r \geq 0. \tag{4.6}$$

Here the precise constant is not important. For $R > 0$ given, we define the rescaled function $\varphi_R : \mathbb{R}^d \to \mathbb{R}$ by

$$\varphi_R(x) = \varphi_R(r) := R^2\varphi(r/R). \tag{4.7}$$

It is easy to see that

$$2 - \varphi''_R(r) \geq 0, \quad 2 - \frac{\varphi'_R(r)}{r} \geq 0, \quad 2d - \Delta\varphi_R(x) \geq 0, \quad \forall r \geq 0, \forall x \in \mathbb{R}^d. \tag{4.8}$$

[1] In [3], the authors assume $u \in C([0,T), H^{2s})$ due to the lack of local theory at that time. Thanks to the local theory given in Section 3, one can recover H^s-valued solutions by an approximation argument (see [3, Section 2]).

Moreover,

$$\|\nabla^j \varphi_R\|_{L^\infty} \lesssim R^{2-j}, \quad j = 0, \cdots, 4,$$

and

$$\operatorname{supp}(\nabla^j \varphi_R) \subset \begin{cases} \{|x| \leq 10R\} & \text{for } j = 1, 2, \\ \{R \leq |x| \leq 10R\} & \text{for } j = 3, 4. \end{cases}$$

By a similar argument as Lemma 2.2 in [3], we have the following virial estimate for the time evolution of $M_{\varphi_R}(u(t))$.

Lemma 4.3 (H^s radial virial estimate)**.** *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $0 < \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$, φ_R be as in (4.7) and $u \in C([0,T), H^s)$ be a radial solution to (1.2). Then for any $t \in [0,T)$,*

$$\begin{aligned} \frac{d}{dt} M_{\varphi_R}(u(t)) &\leq 8s\|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} + \frac{4d\mu_2\alpha_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\quad + O\left(R^{-2s} + R^{-\frac{\alpha_1(d-1)}{2}+\varepsilon_1 s}\|u(t)\|^{\frac{\alpha_1}{2s}+\varepsilon_1}_{\dot{H}^s} + R^{-\frac{\alpha_2(d-1)}{2}+\varepsilon_2 s}\|u(t)\|^{\frac{\alpha_2}{2s}+\varepsilon_2}_{\dot{H}^s}\right) \\ &= 4d\alpha_2 E(u(t)) - 2(d\alpha_2 - 4s)\|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1(\alpha_1-\alpha_2)}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} \\ &\quad + O\left(R^{-2s} + R^{-\frac{\alpha_1(d-1)}{2}+\varepsilon_1 s}\|u(t)\|^{\frac{\alpha_1}{2s}+\varepsilon_1}_{\dot{H}^s} + R^{-\frac{\alpha_2(d-1)}{2}+\varepsilon_2 s}\|u(t)\|^{\frac{\alpha_2}{2s}+\varepsilon_2}_{\dot{H}^s}\right), \end{aligned} \tag{4.9}$$

for any $0 < \varepsilon_1 < \frac{(2d-1)\alpha_1}{2s}$ and any $0 < \varepsilon_2 < \frac{(2d-1)\alpha_2}{2s}$. Here the implicit constant depends only on $\|u_0\|_{L^2}, d, \varepsilon_1, \varepsilon_2, s, \alpha_1$ and α_2.

We also have the following refined version of Lemma 4.3 in the L^2-critical case.

Lemma 4.4 (H^s refined radial virial estimate)**.** *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $0 < \alpha_1 < \alpha_2 = \frac{4s}{d}$, φ_R be as in (4.7) and $u \in C([0,T), H^s)$ be a radial solution to (1.2). Then for any $t \in [0,T)$,*

$$\begin{aligned} \frac{d}{dt} M_{\varphi_R}(u(t)) &\leq 8s\|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} + \frac{4d\mu_2\alpha_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\quad -2\int_0^\infty m^s \int \left(\psi_{1,R} - C_1(\eta)\psi_{2,R}^{\frac{2}{\alpha_1}}\right)|\nabla u_m(t)|^2 dx dm \\ &\quad -2\int_0^\infty m^s \int \left(\psi_{1,R} - C_2(\eta)\psi_{2,R}^{\frac{2}{\alpha_2}}\right)|\nabla u_m(t)|^2 dx dm \\ &\quad + O\Big(R^{-2s} + \eta^{-\beta_1}R^{-\gamma_1} + \eta^{-\beta_2}R^{-\gamma_2} + \eta(1 + R^{-2} + R^{-4})\Big), \end{aligned}$$

for any $\eta > 0$, where

$$\psi_{1,R} := 2 - \varphi_R'', \quad \psi_{2,R} := 2d - \Delta\varphi_R,$$

and $C_1(\eta), C_2(\eta) > 0$,

$$\beta_1 = \frac{\alpha_1}{2-\alpha_1}, \quad \gamma_1 = \frac{\alpha_1(d-2s)}{2-\alpha_1},$$

and similarly for β_2, γ_2. Here the implicit constant depends only on $\|u_0\|_{L^2}, d, s, \alpha_1$ and α_2.

Proof. The proof is essentially given in [3, Lemma 2.3]. For the reader's convenience, we provide some details. By Lemma 4.2,

$$\begin{aligned} \frac{d}{dt} M_\varphi(u(t)) &= -\int_0^\infty m^s \int \Delta^2\varphi |u_m(t)|^2 dx dm + 4\sum_{j,k=1}^d \int_0^\infty m^s \int \partial^2_{jk}\varphi \partial_j \overline{u}_m(t) \partial_k u_m(t) dx dm \\ &\quad + \frac{2\mu_1\alpha_1}{\alpha_1+2}\int \Delta\varphi |u(t)|^{\alpha_1+2} dx + \frac{2\mu_2\alpha_2}{\alpha_2+2}\int \Delta\varphi |u(t)|^{\alpha_2+2} dx. \end{aligned}$$

By Lemma A.2 of [3] and the conservation of mass, we have

$$\left|\int_0^\infty m^s \int \Delta^2\varphi_R |u_m(t)|^2 dx dm\right| \lesssim \|\Delta^2\varphi_R\|^s_{L^\infty}\|\Delta\varphi_R\|^{1-s}_{L^\infty}\|u(t)\|^2_{L^2} \lesssim R^{-2s}.$$

We next use (4.4) to write

$$4\sum_{j,k=1}^{d}\int_0^\infty m^s\int \partial^2_{jk}\varphi_R\partial_j\overline{u}_m(t)\partial_k u_m(t)dxdm = 4\int_0^\infty m^s\int \varphi_R''|\nabla u_m(t)|^2dxdm$$
$$= 8s\|u(t)\|^2_{\dot{H}^s} - 4\int_0^\infty m^s\int(2-\varphi_R'')|\nabla u_m(t)|^2dxdm.$$

We also have

$$\frac{2\mu_1\alpha_1}{\alpha_1+2}\int \Delta\varphi_R|u(t)|^{\alpha_1+2}dx = \frac{4d\mu_1\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} - \frac{2\mu_1\alpha_1}{\alpha_1+2}\int(2d-\Delta\varphi_R)|u(t)|^{\alpha_1+2}dx$$
$$\leq \frac{4d\mu_1\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} + \frac{2|\mu_1|\alpha_1}{\alpha_1+2}\int(2d-\Delta\varphi_R)|u(t)|^{\alpha_1+2}dx.$$

We thus get

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(u(t)) \leq 8s\|u(t)\|^2_{\dot{H}^s} &+ \frac{4d\mu_1\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} + \frac{4d\mu_2\alpha_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} + O(R^{-2s})\\ &-4\int_0^\infty m^s\int \psi_{1,R}|\nabla u_m(t)|^2dxdm \\ &+\frac{2|\mu_1|\alpha_1}{\alpha_1+2}\int\psi_{2,R}|u(t)|^{\alpha_1+2}dx + \frac{2|\mu_2|\alpha_2}{\alpha_2+2}\int\psi_{2,R}|u(t)|^{\alpha_2+2}dx.\end{aligned} \tag{4.10}$$

Since $\mathrm{supp}(\psi_{2,R})\subset\{|x|>R\}$, we use the radial Sobolev embedding (see e.g. [15]):

$$\sup_{x\neq 0}|x|^{\frac{d}{2}-\beta}|u(x)| \leq C(d,\beta)\|u\|_{\dot{H}^\beta}, \quad \frac{1}{2}<\beta<\frac{d}{2} \tag{4.11}$$

and the conservation of mass to estimate

$$\begin{aligned}\int\psi_{2,R}|u(t)|^{\alpha_1+2}dx &= \int_{|x|>R}\left(\psi_{2,R}^{\frac{1}{\alpha_1}}|u(t)|\right)^{\alpha_1}|u(t)|^2dx\\ &\leq \left(\sup_{|x|>R}\psi_{2,R}^{\frac{1}{\alpha_1}}|u(t,x)|\right)^{\alpha_1}\|u(t)\|^2_{L^2}\\ &\lesssim R^{-\frac{\alpha_1(d-2s)}{2}}\left\|(-\Delta)^{s/2}\left(\psi_{2,R}^{\frac{1}{\alpha_1}}u(t)\right)\right\|^{\alpha_1}_{L^2}\\ &\lesssim \eta\left\|(-\Delta)^{s/2}\left(\psi_{2,R}^{\frac{1}{\alpha_1}}u(t)\right)\right\|^2_{L^2} + O\left(\eta^{-\beta_1}R^{-\gamma_1}\right),\end{aligned}$$

where $\beta_1 = \frac{\alpha_1}{2-\alpha_1}$ and $\gamma_1 = \frac{\alpha_1(d-2s)}{2-\alpha_1}$. Here we use the Young inequality $ab \lesssim \eta a^p + \eta^{-q/p}b^q$ with $1/p+1/q=1$ to have the last inequality. Note that the assumption $0<\alpha_1<\alpha_2=\frac{4s}{d}$ and $d>2s$ ensures that $\alpha_1,\alpha_2<2$. By the same argument as in (2.24) of [3], we obtain

$$s\left\|(-\Delta)^{s/2}\left(\psi_{2,R}^{\frac{1}{\alpha_1}}u(t)\right)\right\|^2_{L^2} = \int_0^\infty m^s\int\psi_{2,R}^{\frac{2}{\alpha_1}}|\nabla u_m(t)|^2dxdm + O(1+R^{-2}+R^{-4}).$$

Therefore,

$$\int\psi_{2,R}|u(t)|^{\alpha_1+2}dx = \frac{C\eta}{s}\int_0^\infty m^s\int\chi_{2,R}^{\frac{2}{\alpha_1}}|\nabla u_m(t)|^2dxdm + O(\eta^{-\beta_1}R^{-\gamma_1}+\eta(1+R^{-2}+R^{-4})), \tag{4.12}$$

for some constant $C>0$. The same estimate holds true if α_1 is replaced by α_2. Combining (4.10) and (4.12), we complete the proof. □

4.2. **$\dot{H}^{s_c} \cap \dot{H}^s$ virial estimates.** Let $u \in C([0,T), \dot{H}^{s_c} \cap \dot{H}^s)$ be a solution to (1.2) satisfying

$$\sup_{t \in [0,T)} \|u(t)\|_{\dot{H}^{s_c}} < \infty. \tag{4.13}$$

Recall that s_c is given in (1.3). Let φ_R be as in (4.7). We define the localized virial action

$$M_{\varphi_R}(u(t)) := 2 \int \nabla \varphi_R(x) \cdot \mathrm{Im}\, (\overline{u}(t,x) \nabla u(t,x)) dx.$$

We first note that $M_{\varphi_R}(u(t))$ is well-defined. To see this, we use the Hölder inequality and the Sobolev embedding $\dot{H}^{s_c} \hookrightarrow L^{\alpha_c}$ to have

$$\|u\|_{L^2(|x| \lesssim R)} \lesssim R^{s_c} \|u\|_{L^{\alpha_c}(|x| \lesssim R)} \lesssim R^{s_c} \|u\|_{\dot{H}^{s_c}(|x| \lesssim R)}. \tag{4.14}$$

Here the implicit constant is independent of R. Since $\mathrm{supp}(\nabla \varphi_R) \subset \{|x| \lesssim R\}$, we apply Lemma 4.1 and (4.14) to get

$$\begin{aligned} |M_{\varphi_R}(u(t))| &\leq C(\varphi, R) \Big(\| |\nabla|^{\frac{1}{2}} u(t)\|^2_{L^2(|x| \lesssim R)} + \|u(t)\|_{L^2(|x| \lesssim R)} \| |\nabla|^{\frac{1}{2}} u(t) \|_{L^2(|x| \lesssim R)} \Big) \\ &\leq C(\varphi, R) \Big(\|u(t)\|^{2-\frac{1}{s}}_{L^2(|x| \lesssim R)} \|u(t)\|^{\frac{1}{s}}_{\dot{H}^s(|x| \lesssim R)} + \|u(t)\|^{2-\frac{1}{2s}}_{L^2(|x| \lesssim R)} \|u(t)\|^{\frac{1}{2s}}_{\dot{H}^s(|x| \lesssim R)} \Big) \\ &\leq C(\varphi, R) \Big(\|u(t)\|^{2-\frac{1}{s}}_{\dot{H}^{s_c}(|x| \lesssim R)} \|u(t)\|^{\frac{1}{s}}_{\dot{H}^s(|x| \lesssim R)} + \|u(t)\|^{2-\frac{1}{2s}}_{\dot{H}^{s_c}(|x| \lesssim R)} \|u(t)\|^{\frac{1}{2s}}_{\dot{H}^s(|x| \lesssim R)} \Big) < \infty. \end{aligned} \tag{4.15}$$

Lemma 4.5 ($\dot{H}^{s_c} \cap \dot{H}^s$ radial virial estimate)**.** *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$ and φ_R be as in (4.7). Let $\frac{4s}{d} < \alpha_2 < \frac{4s}{d-2s}$, $\alpha_2 < 4s$, $\frac{d\alpha_2 - 4s}{2s} \leq \alpha_1 < \alpha_2$,*

$$\frac{2s(d\alpha_2 - 4s)}{8s^2 - 4s + (d-2s)\alpha_2} < \alpha_1 < \alpha_2, \tag{4.16}$$

and $u \in C([0,T), \dot{H}^{s_c} \cap \dot{H}^s$ be a radial solution to (1.2) satisfying (4.13). Then for any $t \in [0,T)$ and any $\eta > 0$,

$$\begin{aligned} \frac{d}{dt} M_{\varphi_R}(u(t)) &\leq 8s \|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1\alpha_1}{\alpha_1+2} \|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} + \frac{4d\mu_2\alpha_2}{\alpha_2+2} \|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\quad + O\left(R^{-2(s-s_c)} + C_1(\eta) R^{-\vartheta} + C_2(\eta) R^{-2(s-s_c)} + \eta \|u(t)\|^2_{\dot{H}^s} \right) \\ &= 4d\alpha_2 E(u(t)) - 2(d\alpha_2 - 4s) \|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1(\alpha_1 - \alpha_2)}{\alpha_1+2} \|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} \\ &\quad + O\left(R^{-2(s-s_c)} + C_1(\eta) R^{-\vartheta} + C_2(\eta) R^{-2(s-s_c)} + \eta \|u(t)\|^2_{\dot{H}^s} \right), \end{aligned} \tag{4.17}$$

for some constants $C_1(\eta), C_2(\eta), \vartheta > 0$. Here the implicit constant depends only on $\sup_{t \in [0,T)} \|u(t)\|_{\dot{H}^{s_c}}$, d, s, α_1 and α_2.

Proof. We apply Lemma 4.2 to have

$$\begin{aligned} \frac{d}{dt} M_{\varphi_R}(u(t)) = &- \int_0^\infty m^s \int \Delta^2 \varphi_R |u_m(t)|^2 dx dm + 4 \sum_{j,k} \int_0^\infty m^s \int \partial^2_{jk} \varphi_R \partial_j \overline{u}_m(t) \partial_k u_m(t) dx dm \\ &+ \frac{2\mu_1\alpha_1}{\alpha_1+2} \int \Delta \varphi_R |u(t)|^{\alpha_1+2} dx + \frac{2\mu_2\alpha_2}{\alpha_2+2} \int \Delta \varphi_R |u(t)|^{\alpha_2+2} dx. \end{aligned}$$

Since $\varphi_R(x) = |x|^2$ for $|x| \le R$, we use (4.5) to write

$$\begin{aligned}
\frac{d}{dt} M_{\varphi_R}(u(t)) = 8s\|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} + \frac{4d\mu_2\alpha_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\
-8s\|u(t)\|^2_{\dot{H}^s(|x|>R)} - \frac{4d\mu_1\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}(|x|>R)} - \frac{4d\mu_2\alpha_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}(|x|>R)} \\
- \int_0^\infty m^s \int_{|x|>R} \Delta^2\varphi_R |u_m(t)|^2 dxdm \\
+4\sum_{j,k}\int_0^\infty m^s \int_{|x|>R} \partial^2_{jk}\varphi_R \partial_j \overline{u}_m(t)\partial_k u_m(t) dxdm \\
+\frac{2\mu_1\alpha_1}{\alpha_1+2}\int_{|x|>R}\Delta\varphi_R|u(t)|^{\alpha_1+2}dx + \frac{2\mu_2\alpha_2}{\alpha_2+2}\int_{|x|>R}\Delta\varphi_R|u(t)|^{\alpha_2+2}dx
\end{aligned}$$

$$\begin{aligned}
= 4d\alpha_2 E(u(t)) - 2(d\alpha_2 - 4s)\|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1(\alpha_1-\alpha_2)}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} \\
+4\sum_{j,k}\int_0^\infty m^s \int_{|x|>R}\partial^2_{jk}\varphi_R\partial_j\overline{u}_m(t)\partial_k u_m(t)dxdm - 8s\|u(t)\|^2_{\dot{H}^s(|x|>R)} \\
-\int_0^\infty m^s\int_{|x|>R}\Delta^2\varphi_R|u_m(t)|^2dxdm - \frac{2\mu_1\alpha_1}{\alpha_1+2}\int_{|x|>R}(2d-\Delta\varphi_R)|u(t)|^{\alpha_1+2}dx \\
-\frac{2\mu_2\alpha_2}{\alpha_2+2}\int_{|x|>R}(2d-\Delta\varphi_R)|u(t)|^{\alpha_2+2}dx.
\end{aligned}$$

Using the fact

$$\partial^2_{jk} = \left(\delta_{jk} - \frac{x_j x_k}{r^2}\right)\frac{\partial_r}{r} + \frac{x_j x_k}{r^2}\partial^2_r,$$

and (4.4) and (4.8), we estimate

$$\begin{aligned}
4\sum_{j,k}\int_0^\infty m^s\int_{|x|>R}\partial^2_{jk}\varphi_R\partial_j\overline{u}_m(t)\partial_k u_m(t)dxdm = 4\int_0^\infty m^s\int_{|x|>R}\varphi''_R|\nabla u_m(t)|^2dxdm \\
\le 8s\|u(t)\|^2_{\dot{H}^s(|x|>R)}.
\end{aligned}$$

By Lemma A.2 of [3], the choice of φ_R, we have from (4.14) that

$$\begin{aligned}
\Big|\int_0^\infty m^s\int_{|x|>R}\Delta^2\varphi_R|u_m(t)|^2dxdm\Big| \lesssim \|\Delta^2\varphi_R\|^s_{L^\infty}\|\Delta\varphi_R\|^{1-s}_{L^\infty}\|u\|^2_{L^2(|x|\lesssim R)} \\
\lesssim R^{-2s}R^{2s_c}\|u(t)\|^2_{\dot{H}^{s_c}(|x|\lesssim R)} \lesssim R^{-2(s-s_c)}.
\end{aligned}$$

Collecting above estimates and using $\|2d - \Delta\varphi_R\|_{L^\infty} \lesssim 1$, we get

$$\begin{aligned}
\frac{d}{dt}M_{\varphi_R}(u(t)) \le 4d\alpha_2E(u(t)) - 2(d\alpha_2-4s)\|u(t)\|^2_{\dot{H}^s} + \frac{4d\mu_1(\alpha_1-\alpha_2)}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} \\
+O\left(R^{-2(s-s_c)} + \|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}(|x|>R)} + \|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}(|x|>R)}\right).
\end{aligned} \tag{4.18}$$

Let us now control $\|u(t)\|^{\alpha_j+2}_{L^{\alpha_j+2}(|x|>R)}, j = 1,2$. For H^s solutions, one can take advantage of the mass conservation law. In this setting, the conservation of mass is no longer available. To overcome this difficulty, we use the technique of [39] (see also [21]). Consider for $A > 0$ the annulus $\mathcal{C} := \{A < |x| \le 2A\}$.

Using the radial Sobolev embedding (4.11) and (4.14), we have

$$\begin{aligned}
\|u(t)\|^{\alpha+2}_{L^{\alpha+2}(\mathcal{C})} &\lesssim \left(\sup_{x\in\mathcal{C}} |u(t,x)|\right)^\alpha \|u(t)\|^2_{L^2(\mathcal{C})} \\
&\lesssim A^{-\left(\frac{d}{2}-\beta\right)\alpha} \|u(t)\|^\alpha_{\dot{H}^\beta(\mathcal{C})} \|u(t)\|^2_{L^2(\mathcal{C})} && \left(\frac{1}{2}<\beta<\frac{d}{2}\right) \\
&\lesssim A^{-\left(\frac{d}{2}-\beta\right)\alpha} \left(\|u(t)\|^{\frac{\beta}{s}}_{\dot{H}^s(\mathcal{C})} \|u(t)\|^{1-\frac{\beta}{s}}_{L^2(\mathcal{C})}\right)^\alpha \|u(t)\|^2_{L^2(\mathcal{C})} && \left(\frac{1}{2}<\beta<s<\frac{d}{2}\right) \\
&\lesssim A^{-\left(\frac{d}{2}-\beta\right)\alpha} \|u(t)\|^{\frac{\beta\alpha}{s}}_{\dot{H}^s(\mathcal{C})} \|u(t)\|^{\left(1-\frac{\beta\alpha}{s}\right)\alpha+2}_{L^2(\mathcal{C})} \\
&\lesssim A^{-\vartheta} \|u(t)\|^{\frac{\beta\alpha}{s}}_{\dot{H}^s(\mathcal{C})},
\end{aligned}$$

where

$$\vartheta := \left(\frac{d}{2}-\beta\right)\alpha - \left(\left(1-\frac{\beta\alpha}{s}\right)\alpha+2\right)s_{\rm c}.$$

In the case $\alpha=\alpha_2$, we see that

$$\vartheta=\vartheta_2 = 2(s-s_{\rm c})\left(1-\frac{\beta_2\alpha_2}{2s}\right).$$

Thanks to the assumption $\alpha_2<4s$, we can choose $\frac{1}{2}<\beta_2<s$ so that $\frac{\beta_2\alpha_2}{s}<2$. The Young inequality then implies for any $\eta>0$,

$$A^{-\vartheta_2}\|u(t)\|^{\frac{\beta_2\alpha_2}{s}}_{\dot{H}^s(\mathcal{C})} \lesssim \eta\|u(t)\|^2_{\dot{H}^s(\mathcal{C})} + C_2(\eta)A^{-\frac{2s\vartheta_2}{2s-\beta_2\alpha_2}} = \eta\|u(t)\|^2_{\dot{H}^s(\mathcal{C})} + C_2(\eta)A^{-2(s-s_{\rm c})}.$$

This shows for any $\eta>0$, there exists $C_2(\eta)>0$ such that

$$\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}(\mathcal{C})} \lesssim \eta\|u(t)\|^2_{\dot{H}^s(\mathcal{C})} + C_2(\eta)A^{-2(s-s_{\rm c})}. \tag{4.19}$$

In the case $\alpha=\alpha_1$, we have

$$\vartheta=\vartheta_1 = 2(s-s_{\rm c})\left(1-\frac{\beta_1\alpha_1}{2s}\right) - 2s\left(1-\frac{\alpha_1}{\alpha_2}\right).$$

Applying the Young inequality, we get for any $\eta>0$,

$$A^{-\vartheta_1}\|u(t)\|^{\frac{\beta_1\alpha_1}{s}}_{\dot{H}^s(\mathcal{C})} \lesssim \eta\|u(t)\|^2_{\dot{H}^s(\mathcal{C})} + C_1(\eta)A^{-\frac{2s\vartheta_1}{2s-\beta_1\alpha_1}} = \eta\|u(t)\|^2_{\dot{H}^s(\mathcal{C})} + C_1(\eta)A^{-\tilde{\vartheta}_1},$$

where

$$\tilde{\vartheta}_1 := \frac{2s\vartheta_1}{2s-\beta_1\alpha_1} = 2(s-s_{\rm c}) - \frac{4s^2}{2s-\beta_1\alpha_1}\left(1-\frac{\alpha_1}{\alpha_2}\right).$$

We need to show $\tilde{\vartheta}_1>0$. Since $\alpha_1<\alpha_2<4s$, taking $\beta_1=\frac{1}{2}+\epsilon$ for some $\epsilon>0$ small enough, we see that $\tilde{\vartheta}_1>0$ provided that

$$2(s-s_{\rm c}) - \frac{8s^2}{4s-\alpha_1}\left(1-\frac{\alpha_1}{\alpha_2}\right)>0.$$

It is easy to check that the assumption (4.16) ensures that the above inequality holds true. This shows that for any $\eta>0$, there exists $C_1(\eta)>0$ such that

$$\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}(\mathcal{C})} \lesssim \eta\|u(t)\|^2_{\dot{H}^s(\mathcal{C})} + C_1(\eta)A^{-\tilde{\vartheta}_1}, \tag{4.20}$$

for some $\tilde{\vartheta}_1>0$. In both cases, we have shown

$$\|u(t)\|^{\alpha+2}_{L^{\alpha+2}(\mathcal{C})} \lesssim \eta\|u(t)\|^2_{\dot{H}^s(\mathcal{C})} + C(\eta)A^{-\vartheta}, \tag{4.21}$$

for some $\vartheta>0$. We now write

$$\|u(t)\|^{\alpha+2}_{L^{\alpha+2}(|x|>R)} = \sum_{j=0}^\infty \int_{2^jR<|x|\leq 2^{j+1}R} |u(t)|^{\alpha+2}dx,$$

and apply (4.21) with $A = 2^j R$ to get

$$\int_{|x|>R} |u(t)|^{\alpha+2}dx \lesssim \eta \sum_{j=0}^{\infty} \|u(t)\|^2_{\dot H^s(2^jR<|x|\leq 2^{j+1}R)} + C(\eta)\sum_{j=0}^{\infty}(2^jR)^{-\vartheta} \tag{4.22}$$

$$\lesssim \eta\|u(t)\|^2_{\dot H^s(|x|>R)} + C(\eta)R^{-\vartheta}. \tag{4.23}$$

This estimate combined with (4.18), (4.19) and (4.20) show (4.17). The proof is complete. □

5. L^2-Stability

In this section, we study the stability of the L^2-critical fractional nonlinear Schrödinger equation. This stability is very useful to study the energy scattering of (1.2). Let us start by introducing some notations. For any spacetime slab $J \times \mathbb{R}^d$, the Strichartz norm $\dot F^0(J)$ is defined by

$$\|u\|_{\dot F^0(J)} := \sup\left\{\|u\|_{L^p(J,L^q)} \ : \ (p,q) \text{ is fractional admissible}\right\}.$$

The Strichartz norm $\dot F^s(J)$ is defined by

$$\|u\|_{\dot F^s(J)} := \||\nabla|^s u\|_{\dot F^0(J)}.$$

Let $\dot N^0(J)$ be the dual space of $\dot F^0(J)$, i.e.

$$\|u\|_{\dot N^0(J)} := \inf\left\{\|u\|_{L^{a'}(J,L^{b'})} \ : \ (a,b) \text{ is fractional admissible}\right\}.$$

We define

$$\dot N^s(J) := \left\{u \ : \ |\nabla|^s u \in \dot N^0(J)\right\}.$$

Denote also

$$\|u\|_{V(J)} := \|u\|_{L^{\frac{2(d+2s)}{d}}(J\times\mathbb{R}^d)}, \tag{5.1}$$

where $\left(\frac{2(d+2s)}{d}, \frac{2(d+2s)}{d}\right)$ is a fractional admissible pair.

Consider the defocusing L^2-critical fractional nonlinear Schrödinger equation

$$\left\{\begin{array}{rcl} i\partial_t v - (-\Delta)^s v &=& |v|^{\frac{4s}{d}}v, \quad \text{on } \mathbb{R}^+\times\mathbb{R}^d, \\ v(0) &=& v_0. \end{array}\right. \tag{5.2}$$

To study the stability of (5.2), we assume that Assumption 1.1 holds true.

Proposition 5.1 (L^2-critical stability)**.** *Let $d \geq 2$ and $\frac{d}{2d-1} \leq s < 1$. Let $J \subset \mathbb{R}^+$ be a compact interval and let $\tilde v$ be a radial approximate solution to (5.2) in the sense that*

$$i\partial_t \tilde v - (-\Delta)^s \tilde v = |\tilde v|^{\frac{4s}{d}}\tilde v + e, \tag{5.3}$$

for some function e. Assume that

$$\|\tilde v\|_{L^\infty(J,L^2)} \leq M, \tag{5.4}$$

$$\|\tilde v\|_{V(J)} \leq L, \tag{5.5}$$

for some constants $M, L > 0$. Let $t_0 \in J$ and let $v(t_0)$ be radially symmetric and close to $\tilde v(t_0)$ in the sense that

$$\|v(t_0) - \tilde v(t_0)\|_{L^2} \leq M', \tag{5.6}$$

for some $M' > 0$. Assume in addition that

$$\|e^{-(t-t_0)(-\Delta)^s}(v(t_0) - \tilde v(t_0))\|_{V(J)} \leq \epsilon, \tag{5.7}$$

$$\|e\|_{\dot N^0(J)} \leq \epsilon, \tag{5.8}$$

for some $0 < \epsilon < \epsilon_0$ where $\epsilon_0 = \epsilon_0(M, L, M') > 0$ is a small constant. Then there exists a solution v to (5.2) on $J \times \mathbb{R}^d$ with intial data $v(t_0)$ at time $t = t_0$ satisfying

$$\|v - \tilde{v}\|_{V(J)} \leq \epsilon C(M, L, M'), \tag{5.9}$$

$$\|v - \tilde{v}\|_{\dot{F}^0(J)} \leq C(M, L, M')M', \tag{5.10}$$

$$\|v\|_{\dot{F}^0(J)} \leq C(M, L, M'). \tag{5.11}$$

The proof of this result follows easily from the same lines as in [44, Lemma 3.6]. We thus omit the proof.

Remark 5.2. Assumption 1.1 ensures (5.5) to hold. Moreover, since $\left(\frac{2(d+2s)}{d}, \frac{2(d+2s)}{d}\right)$ is fractional admissible, it follows from Strichartz estimates that the condition (5.6) is redundant if $M' = O(\epsilon)$.

Lemma 5.3. *Let $J \subset \mathbb{R}^+$ be an interval and let v be a unique solution to (5.2) on $J \times \mathbb{R}^d$ satisfying*

$$\|v\|_{V(J)} \leq L, \tag{5.12}$$

then if $t_0 \in J$ and for $\gamma = 0$ or $\gamma = s$, $v(t_0) \in H^\gamma$, then

$$\|v\|_{\dot{F}^\gamma(J)} \leq C(L)\|v(t_0)\|_{\dot{H}^\gamma}. \tag{5.13}$$

Proof. Dividing J into $K \sim \left(1 + \frac{L}{\eta}\right)^{\frac{2(d+2s)}{d}}$ subintervals $J_k = [t_k, t_{k+1}]$ such that

$$\|v\|_{V(J_k)} \leq \eta, \quad \forall k = 0, \cdots K - 1,$$

for some $\eta > 0$ small enough to be chosen shortly. By Strichartz estimates,

$$\begin{aligned}
\|v\|_{\dot{F}^\gamma(J_k)} &\lesssim \|v(t_k)\|_{\dot{H}^\gamma} + \||\nabla|^\gamma(|v|^{\frac{4s}{d}}v)\|_{L^{\frac{2(d+2s)}{d+4s}}(J_k \times \mathbb{R}^d)} \\
&\lesssim \|v(t_k)\|_{\dot{H}^\gamma} + \|v\|^{\frac{4s}{d}}_{V(J_k)}\||\nabla|^\gamma v\|_{V(J_k)} \\
&\lesssim \|v(t_k)\|_{\dot{H}^\gamma} + \eta^{\frac{4s}{d}}\|v\|_{\dot{F}^\gamma(J_k)}.
\end{aligned}$$

Choosing $\eta > 0$ small enough, we get $\|v\|_{\dot{F}^\gamma(J_k)} \lesssim \|v(t_j)\|_{\dot{H}^\gamma}$. By induction, we see that $\|v\|_{\dot{F}^\gamma(J_k)} \lesssim \|v(t_0)\|_{\dot{H}^\gamma}$ for all $k = 0, \cdots, K - 1$. Summing over all subintervals J_k, we prove (5.13). □

Remark 5.4. It follows from Lemma 5.3 and Assumption 1.1 that the global solution to (5.2) satisfies

$$\|v\|_{\dot{F}^0(\mathbb{R}^+)} \leq C(\|v_0\|_{L^2}).$$

6. Global well-posedness

Theorem 6.1 (Global well-posedness). *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$. Let $u_0 \in H^s$ be radial. Then there exists a unique global solution to (1.2) if one of the following conditions holds true:*

- $0 < \alpha_1 < \alpha_2 < \frac{4s}{d}$ *and* $\mu_1, \mu_2 \in \mathbb{R}$;
- $0 < \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$ *and* $\mu_1 \in \mathbb{R}, \mu_2 > 0$.

Moreover, for all compact intervals $J \subset \mathbb{R}^+$, the global solution satisfies the spacetime bound

$$\|u\|_{F^s(J)} \leq C(|J|, \|u_0\|_{H^s}). \tag{6.1}$$

Here for simplicity, we only state the global well-posedness for radial H^s data. However, it still holds true for non-radial H^s data provided the local theory is available (see e.g. Proposition 3.1).

Proof. The proof of this result is based on the blow-up alternative which asserts that the time of existence depends only on the H^s-norm of initial data, and an a priori estimate on the kinetic energy, namely

$$\sup_{t\in[0,T)} \|u(t)\|_{\dot{H}^s} \leq C(E,M), \tag{6.2}$$

where E and M are the conserved energy and mass respectively. To prove (6.2), we consider the following three cases:

- When μ_1 and μ_2 are both positive, the conservation of energy gives obviously

$$\sup_{t\in[0,T)} \|u(t)\|^2_{\dot{H}^s} \lesssim E.$$

- When $\mu_1 < 0$ and $\mu_2 > 0$, we use the following inequality

$$\frac{\mu_1}{\alpha_1+2}|u|^{\alpha_1+2} + \frac{\mu_2}{\alpha_2+2}|u|^{\alpha_2+2} \geq -C(\mu_1,\mu_2,\alpha_1,\alpha_2)|u|^2, \tag{6.3}$$

together with the conservation of mass and energy to have

$$\sup_{t\in[0,T)} \|u(t)\|^2_{\dot{H}^s} \lesssim E+M.$$

To see (6.3), we use the Young inequality to have for any $\eta>0$,

$$|u|^{\alpha_1+2} \leq \eta |u|^{\alpha_2+2} + C(\eta)|u|^2.$$

Multiplying both sides by $\frac{\mu_1}{\alpha_1+2}$, we get

$$\frac{\mu_1}{\alpha_1+2}|u|^{\alpha_1+2} \geq \frac{\eta\mu_1}{\alpha_1+2}|u|^{\alpha_2+2} + \frac{C(\eta)\mu_1}{\alpha_1+2}|u|^2.$$

We next choose $\eta>0$ so that $\frac{\mu_1\eta}{\alpha_1+2} = \frac{-\mu_2}{\alpha_2+2}$ or $\eta = -\frac{\mu_2(\alpha_1+2)}{\mu_1(\alpha_2+2)} > 0$ and obtain (6.3).

- When both μ_1 and μ_2 are negative, the hypotheses force $0<\alpha_1<\alpha_2<\frac{4s}{d}$. By Gagliardo-Nirenberg inequality, we have for any $t\in[0,T)$,

$$\begin{aligned}\|u(t)\|^{\alpha_j+2}_{L^{\alpha_j+2}} &\lesssim \|u(t)\|^{\alpha_j+2-\frac{d\alpha_j}{2s}}_{L^2} \|u(t)\|^{\frac{d\alpha_j}{2s}}_{\dot{H}^s} \\ &\lesssim M^{\frac{\alpha_j+2}{2}-\frac{d\alpha_j}{4s}} \|u(t)\|^{\frac{d\alpha_j}{2s}}_{\dot{H}^s}.\end{aligned}$$

We next apply the Young inequality

$$ab \lesssim \eta a^p + \eta^{-\frac{q}{p}} b^q, \quad a,b,\eta>0, \quad 1<p,q<\infty, \quad \frac{1}{p}+\frac{1}{q}=1$$

with $a = \|u(t)\|^{\frac{d\alpha_j}{2s}}_{\dot{H}^s}$, $b=1$ and $p=\frac{4s}{d\alpha_j}$ to get

$$\|u(t)\|^{\alpha_j+2}_{L^{\alpha_j+2}} \lesssim M^{1-\frac{(d-2s)\alpha_j}{4s}} \left(\eta\|u(t)\|^2_{\dot{H}^s} + \eta^{-\frac{d\alpha_j}{4s-d\alpha_j}}\right).$$

Taking $\eta = c^2 M^{\frac{(d-2s)\alpha_j}{2s}-2}$ for $0<c\ll 1$, we obtain

$$\|u(t)\|^{\alpha_j+2}_{L^{\alpha_j+2}} \leq c\|u(t)\|^2_{\dot{H}^s} + C(M),$$

for $j=1,2$. The conservation of energy then implies

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2_{\dot{H}^s} &= E - \frac{\mu_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} - \frac{\mu_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\leq E - \left(\frac{\mu_1}{\alpha_1+2}+\frac{\mu_2}{\alpha_2+2}\right)(c\|u(t)\|^2_{\dot{H}^s} + C(M)).\end{aligned}$$

Taking $c>0$ sufficiently small, we absorb $\|u(t)\|^2_{\dot{H}^s}$ in the right hand side to the left hand side and get

$$\sup_{t\in[0,T)} \|u(t)\|_{\dot{H}^s} \leq C(E,M).$$

Combining three cases, we prove (6.2). The proof is complete. □

7. Scattering

In this section, we show the asymptotic behavior in the energy space H^s for (1.2). More precisely, we prove the following.

Theorem 7.1 (Energy scattering). *Let $d \geq 3$, $\frac{d}{2d-1} \leq s < 1$, $\frac{4s}{d} \leq \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$ and $u_0 \in H^s$ be radial. If $\alpha_1 = \frac{4s}{d}$, then we also assume Assumption 1.1. Then the corresponding global solution u to (1.2) scatters in H^s forward in time, i.e. there exists $u_0^+ \in H^s$ such that*

$$\|u(t) - e^{-it(-\Delta)^s}u_0^+\|_{H^s} \to 0,$$

as $t \to +\infty$ if one of the following conditions holds:

- *$\mu_1, \mu_2 > 0$;*
- *$\mu_1 < 0, \mu_2 > 0$ and $M \leq c(\|u_0\|_{\dot{H}^s})$ for some small constant c depending only on $\|u_0\|_{\dot{H}^s}$.*

The proof of this result is based on the combination of the L^2-stability and the radial Morawetz inequality. We will give the proof of Theorem 7.1 at the end of this section.

7.1. Radial Morawetz inequality.

Lemma 7.2 (Radial Morawetz inequality). *Let $d \geq 3$, $\frac{d}{2d-1} \leq s < 1$, $\frac{4s}{d} \leq \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$, $\mu_1, \mu_2 > 0$ and $u_0 \in H^s$ be radial. Then the corresponding global solution u to (1.2) satisfies the global bound*

$$\|u\|_{L^m(\mathbb{R}\times\mathbb{R}^d)} \leq C(E, M),$$

for any $\alpha_1 + 1 + \frac{2}{d-2s} \leq m \leq \alpha_2 + 1 + \frac{2}{d-2s}$.

Proof. We firstly note that under the assumptions of Lemma 7.2, the solution exists globally in time due to Section 6. Consider $\varphi(x) = |x|$. A direct computation shows

$$\partial_j\varphi(x) = \frac{x_j}{|x|}, \quad \partial^2_{jk}\varphi(x) = \frac{1}{|x|}\left(\delta_{jk} - \frac{x_jx_k}{|x|^2}\right), \quad j,k = 1,\cdots,d.$$

In particular, $\nabla\varphi(x) = \frac{x}{|x|}$, $\quad \Delta\varphi(x) = \frac{d-1}{|x|}$. Moreover,

$$-\Delta^2\varphi(x) = -(d-1)\Delta\left(\frac{1}{|x|}\right) = \begin{cases} 4\pi(d-1)\delta_0 & \text{if } d = 3, \\ \frac{(d-1)(d-3)}{|x|^3} & \text{if } d \geq 4, \end{cases}$$

where δ_0 is the Dirac delta function. Note also that since φ is a convex function, it is well-known that

$$\sum_{j,k}\partial^2_{jk}\varphi \mathrm{Re}\,(\partial_j\overline{u}\partial_k u) \geq 0.$$

Applying formally Lemma 4.2 with $\varphi(x) = |x|$, we obtain

$$\sum_{j=1}^2 \frac{2\alpha_j(d-1)}{\alpha_j+2}\int\frac{|u(t,x)|^{\alpha_j+1}}{|x|}dx \leq \frac{d}{dt}M_{|x|}(u(t)).$$

Taking the time integration, we have

$$\sum_{j=1}^2\iint_{\mathbb{R}\times\mathbb{R}^d}\frac{|u(t,x)|^{\alpha_j+1}}{|x|}dxdt \lesssim \sup_{t\in\mathbb{R}} M_{|x|}(u(t)).$$

Recall (see [18, Lemma 2.3]) that $|M_{|x|}(u(t))| \lesssim \|u(t)\|^2_{\dot{H}^{1/2}} \lesssim \|u(t)\|^2_{H^s}$. Therefore,

$$\sum_{j=1}^2\iint_{\mathbb{R}\times\mathbb{R}^d}\frac{|u(t,x)|^{\alpha_j+1}}{|x|}dxdt \leq C(E,M). \tag{7.1}$$

We next apply the radial Sobolev embedding

$$|x|^{\frac{d-2s}{2}}|u(x)| \lesssim \|u(t)\|_{\dot{H}^s}. \tag{7.2}$$

Combining (7.1) and (7.2), we have

$$\sum_{j=1}^{2} \iint_{\mathbb{R}\times\mathbb{R}^d} |u(t,x)|^{\alpha_j+1+\frac{2}{d-2s}} dxdt \le C(E,M).$$

Interpolating between $L^{\alpha_j+1+\frac{2}{d-2s}}$, $j=1,2$, we complete the proof. □

7.2. **Global Strichartz bound.** In this subsection, we use the radial Morawetz bound given in Lemma 7.2 to show the global Strichartz bound for (1.2), namely

$$\|u\|_{F^s(\mathbb{R}^+)} \le C(E,M). \tag{7.3}$$

Let us start with the following useful estimate.

Lemma 7.3. *Let $d\ge 2$, $0<s<1$, $\frac{4s}{d}<\alpha<\frac{4s}{d-2s}$, $1<m<\infty$ and $\gamma=0$ or $\gamma=s$. Then there exists $\epsilon>0$ small enough such that for any spacetime slab $J\times\mathbb{R}^d$,*

$$\||\nabla|^\gamma(|u|^\alpha u)\|_{\dot{N}^0(J)} \lesssim \||\nabla|^\gamma u\|_{L^{2+\epsilon}(J,L^{\frac{2d(2+\epsilon)}{d(2+\epsilon)-4s}})} \|u\|^{\frac{m\epsilon}{2(2+\epsilon)}}_{L^m(J\times\mathbb{R}^d)} \|u\|^{a(\epsilon)}_{L^\infty(J,L^2)} \|u\|^{b(\epsilon)}_{L^\infty(J,\dot{H}^s)}, \tag{7.4}$$

where

$$a(\epsilon) := \alpha\left(1-\frac{d}{2s}\right) + \frac{16s-(d-2s)(2-m)\epsilon}{4s(2+\epsilon)}, \quad b(\epsilon) := \frac{d\alpha}{2s} - \frac{16s+((m-2)d+4s)\epsilon}{4s(2+\epsilon)}.$$

Proof. By Hölder's inequality and fractional derivative estimates, we estimate

$$\||\nabla|^\gamma(|u|^\alpha u)\|_{\dot{N}^0(J)} \le \||\nabla|^\gamma(|u|^\alpha u)\|_{L^2(J,L^{\frac{2d}{d+2s}})} \lesssim \||\nabla|^\gamma u\|_{L^{2+\epsilon}(J,L^{\frac{2d(2+\epsilon)}{d(2+\epsilon)-4s}})} \|u\|^\alpha_{L^{\frac{2\alpha(2+\epsilon)}{\epsilon}}(J,L^{\frac{d\alpha(2+\epsilon)}{s(4+\epsilon)}})}.$$

We next bound

$$\|u\|_{L^{\frac{2\alpha(2+\epsilon)}{\epsilon}}(J,L^{\frac{d\alpha(2+\epsilon)}{s(4+\epsilon)}})} \lesssim \|u\|^{\theta_1}_{L^m(J\times\mathbb{R}^d)} \|u\|^{1-\theta_1}_{L^\infty(J,L^a)},$$

with

$$\theta_1 = \frac{m\epsilon}{2\alpha(2+\epsilon)}, \quad \frac{1-\theta_1}{a} = \frac{8s-(d-2s)\epsilon}{2d\alpha(2+\epsilon)}.$$

We next use the Hölder inequality and the Sobolev embedding to have

$$\|u\|_{L^\infty(J,L^a)} \lesssim \|u\|^{\theta_2}_{L^\infty(J,L^2)} \|u\|^{1-\theta_2}_{L^\infty(J,L^{\frac{2d}{d-2s}})} \lesssim \|u\|^{\theta_2}_{L^\infty(J,L^2)} \|u\|^{1-\theta_2}_{L^\infty(J,\dot{H}^s)},$$

provided that

$$\frac{1}{a} = \frac{\theta_2}{2} + \frac{(1-\theta_2)(d-2s)}{2d}.$$

We thus obtain

$$\||\nabla|^\gamma(|u|^\alpha u)\|_{\dot{N}^0(J)} \lesssim \||\nabla|^\gamma u\|_{L^{2+\epsilon}(J,L^{\frac{2d(2+\epsilon)}{d(2+\epsilon)-4s}})} \|u\|^{\alpha\theta_1}_{L^m(J\times\mathbb{R}^d)} \|u\|^{\alpha(1-\theta_1)\theta_2}_{L^\infty(J,L^2)} \|u\|^{\alpha(1-\theta_1)(1-\theta_2)}_{L^\infty(J,\dot{H}^s)}.$$

This shows (7.4) with

$$a(\epsilon) := \alpha(1-\theta_1)\theta_2 = \alpha\left(1-\frac{d}{2s}\right) + \frac{16s-(d-2s)(2-m)\epsilon}{4s(2+\epsilon)},$$
$$b(\epsilon) := \alpha(1-\theta_1)(1-\theta_2) = \frac{d\alpha}{2s} - \frac{16s+((m-2)d+4s)\epsilon}{4s(2+\epsilon)}.$$

In order to perform the above estimates, we need to show $a(\epsilon)$ and $b(\epsilon)$ are both positive for $\epsilon>0$ small enough. Since $\epsilon\mapsto a(\epsilon)$ and $\epsilon\mapsto b(\epsilon)$ are continuous functions and the limits

$$\lim_{\epsilon\to 0} a(\epsilon) = 2-\frac{\alpha(d-2s)}{2s}, \quad \lim_{\epsilon\to 0} b(\epsilon) = \frac{d\alpha}{2s} - 2$$

are both positive since $\frac{4s}{d} < \alpha < \frac{4s}{d-2s}$. Taking $\epsilon > 0$ small enough, we see that $a(\epsilon), b(\epsilon) > 0$. The proof is complete. □

For any spacetime slab $J \times \mathbb{R}^d$, we denote

$$\|u\|_{W(J)} := \|u\|_{L^{\frac{2(d+2s)}{d-2s}}(J, L^{\frac{2d(d+2s)}{d^2+4s^2}})}, \tag{7.5}$$

where $\left(\frac{2(d+2s)}{d-2s}, \frac{2d(d+2s)}{d^2+4s^2}\right)$ is a fractional admissible pair. The following estimate is useful to show the energy scattering for (1.2).

Lemma 7.4. *Let $d \geq 2$, $0 < s < 1$, $\frac{4s}{d} \leq \alpha < \frac{4s}{d-2s}$ and $\gamma = 0$ or $\gamma = s$. Then*

$$\||\nabla|^\gamma (|u|^\alpha u)\|_{\dot{N}^0(J)} \lesssim \|u\|_{V(J)}^{2-\frac{(d-2s)\alpha}{2s}} \||\nabla|^s u\|_{W(J)}^{\frac{d\alpha}{2s}-2} \||\nabla|^\gamma u\|_{V(J)}, \tag{7.6}$$

where the V-norm is given in (5.1).

Proof. By Hölder's inequality and fractional derivative estimates, we have

$$\begin{aligned}
\||\nabla|^\gamma (|u|^\alpha u)\|_{\dot{N}^0(J)} &\leq \||\nabla|^\gamma (|u|^\alpha u)\|_{L^{\frac{2(d+2s)}{d+4s}}(J\times\mathbb{R}^d)} \\
&\lesssim \||u|^\alpha\|_{L^{\frac{d+2s}{2s}}(J\times\mathbb{R}^d)} \||\nabla|^\gamma u\|_{L^{\frac{2(d+2s)}{d}}(J\times\mathbb{R}^d)} \\
&\lesssim \|u\|^\alpha_{L^{\frac{(d+2s)\alpha}{2s}}(J\times\mathbb{R}^d)} \||\nabla|^\gamma u\|_{L^{\frac{2(d+2s)}{d}}(J\times\mathbb{R}^d)} \\
&\lesssim \|u\|^{2-\frac{(d-2s)\alpha}{2s}}_{L^{\frac{2(d+2s)}{d}}(J\times\mathbb{R}^d)} \|u\|^{\frac{d\alpha}{2s}-2}_{L^{\frac{2(d+2s)}{d-2s}}(J\times\mathbb{R}^d)} \||\nabla|^\gamma u\|_{L^{\frac{2(d+2s)}{d}}(J\times\mathbb{R}^d)}.
\end{aligned}$$

Using the Sobolev embedding

$$\|u\|^{\frac{d\alpha}{2s}-2}_{L^{\frac{2(d+2s)}{d-2s}}(J\times\mathbb{R}^d)} \lesssim \||\nabla|^s u\|_{L^{\frac{2(d+2s)}{d-2s}}(J, L^{\frac{2d(d+2s)}{d^2+4s^2}})},$$

we prove the desired estimate. □

We are now able to show global Strichartz bounds (7.3) for solutions to (1.2). We will consider three cases.

(1) The case $\frac{4s}{d} = \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$ and $\mu_1, \mu_2 > 0$. Without loss of generality, we assume that $\mu_1 = \mu_2 = 1$. In this case, we view (1.2) as a perturbation to the L^2-critical fractional nonlinear Schrödinger equation. Note that in this case, we need to assume a satisfactory global theory for the L^2-critical problem, more precisely Assumption 1.1.

Fix $m \in \left[\alpha_1 + 1 + \frac{2}{d-2s}, \alpha_2 + 1 + \frac{2}{d-2s}\right]$. By Lemma 7.2, we have

$$\|u\|_{L^m(\mathbb{R}^+\times\mathbb{R}^d)} \leq C(E, M).$$

Let $\eta > 0$ be a small constant to be chosen later. We split $\mathbb{R}^+$ into N subintervals I_n such that

$$\|u\|_{L^m(I_n\times\mathbb{R}^d)} \sim \eta, \quad \forall n = 0, \cdots, N-1.$$

We will show that on each spacetime slab $I_n \times \mathbb{R}^d$, $n = 0, \cdots, N-1$, the solution u obeys

$$\|u\|_{F^s(I_n)} \leq C(E, M). \tag{7.7}$$

Summing these bounds over all subintervals $I_n, n = 0, \cdots, N-1$, we obtain (7.3).

Let $\epsilon > 0$ be a small constant such that (7.4) holds. Denote

$$\dot{X}^0(I_n) := L^{2+\epsilon}(I_n, L^{\frac{2d(2+\epsilon)}{d(2+\epsilon)-4s}}) \cap V(I_n),$$

where the V-norm is introduced in (5.1). We have from Lemma 7.3 and the conservation of mass and energy that

$$\begin{aligned}\||u|^{\alpha_2}u\|_{\dot N^0(I_n)} &\lesssim \|u\|_{L^{2+\epsilon}(I_n, L^{\frac{2d(2+\epsilon)}{d(2+\epsilon)-4s}})}\|u\|^{\frac{m\epsilon}{2(2+\epsilon)}}_{L^m(I_n\times\mathbb{R}^d)}\|u\|^{a(\epsilon)+b(\epsilon)}_{L^\infty(I_n,H^s)} \\ &\lesssim C(E,M)\eta^c\|u\|_{\dot X^0(I_n)},\end{aligned} \tag{7.8}$$

where $c := \frac{m\epsilon}{2(2+\epsilon)} > 0$ is a small constant.

Let us fix $I_n = [a,b]$. As mentioned before, we view the solution u defined on $I_n\times\mathbb{R}^d$ as a perturbation to the solution of the L^2-critical fractional NLS equation

$$\begin{cases} i\partial_t v - (-\Delta)^s v &= |v|^{\frac{4s}{d}}v, \\ v(a) &= u(a). \end{cases}$$

By Assumption 1.1, this initial value problem is globally well-posed in H^s and the global solution satisfies

$$\|v\|_{V(\mathbb{R}^+)} \leq C(\|v(a)\|_{L^2}) = C(M).$$

We have from Remark 5.4 that

$$\|v\|_{\dot F^0(\mathbb{R}^+)} \leq C(M).$$

Using this global bound and the fact $\dot F^0(\mathbb{R}^+)\subset \dot X^0(\mathbb{R}^+)$, we split $\mathbb{R}^+$ into K subintervals J_k such that

$$\|v\|_{\dot X^0(J_k)} \sim \delta, \quad \forall k = 0,\cdots,K-1, \tag{7.9}$$

where $\delta > 0$ is a small constant to be chosen later.

We are only interested in those intervals $J_k = [t_k, t_{k+1}]$ which have a nonempty intersection with I_n. Without loss of generality, we may assume that

$$I_n = \cup_{k=0}^{L-1} J_k, \quad t_0 = a, \quad t_L = b.$$

Since the nonlinear evolution v is small on $J_k\times\mathbb{R}^d$ (see (7.9)), the linear evolution $e^{-i(t-t_k)(-\Delta)^s}v(t_k)$ is also small on $J_k\times\mathbb{R}^d$. Indeed, by Strichartz estimates and (7.9),

$$\begin{aligned}\|e^{-i(t-t_k)(-\Delta)^s}v(t_k)\|_{\dot X^0(J_k)} &\leq \|v\|_{\dot X^0(J_k)} + C\||v|^{\frac{4s}{d}}v\|_{L^{p'}(J_k\times\mathbb{R}^d)} \\ &\leq \delta + C\|v\|^{\frac{4s}{d}+1}_{V(J_k)} \leq \delta + C\delta^{\frac{4s}{d}+1},\end{aligned}$$

where p' is the conjugate exponent of $p = \frac{2(d+2s)}{d}$. Note that (p,p) is fractional admissible and

$$\frac{1}{p'} = \frac{4s}{dp} + \frac{1}{p}.$$

Choosing $\delta > 0$ small enough, we obtain

$$\|e^{-i(t-t_k)(-\Delta)^s}v(t_k)\|_{\dot X^0(J_k)} \leq 2\delta. \tag{7.10}$$

In order to estimate F^s-norm of u on $I_n\times\mathbb{R}^d$, we use the the stability technique as follows. We firstly compare u to v on the slab $[t_0,t_1]\times\mathbb{R}^d$ by the L^2-stability (see Proposition 5.1). We then use the result as an input to compare u to v on the slab $[t_1,t_2]\times\mathbb{R}^d$. By induction, we derive bounds on u from bounds on v on all slab $J_k\times\mathbb{R}^d$, $k=0,\cdots,L$. Adding these bounds, we get the desired estimate of u on $I_n\times\mathbb{R}^d$.

For $k=0$, we will check the hypotheses of Proposition 5.1. Note that u and $|u|^{\alpha_2}u$ play the roles of $\tilde v$ and e in (5.3) respectively. The condition (5.4) is satisfied by the conservation of mass. The conditions (5.6) and (5.7) are obvious since $u(t_0) = v(t_0)$. It remains to check (5.5) and (5.8). Using Duhamel's formula

$$u(t) = e^{-i(t-t_0)(-\Delta)^s}u(t_0) - i\int_{t_0}^t e^{-i(t-\tau)(-\Delta)^s}(|u|^{\frac{4s}{d}}u)(\tau)d\tau - i\int_{t_0}^t e^{-i(t-\tau)(-\Delta)^s}(|u|^{\alpha_2}u)(\tau)d\tau,$$

Strichartz estimates, (7.8) and (7.10) imply

$$\begin{aligned}\|u\|_{\dot{X}^0(J_0)} &\leq \|e^{-i(t-t_0)(-\Delta)^s}u(t_0)\|_{\dot{X}^0(J_0)} + C\||u|^{\frac{4s}{d}}u\|_{\dot{N}^0(J_0)} + C\||u|^{\alpha_2}u\|_{\dot{N}^0(J_0)} \\ &\leq 2\delta + C\|u\|^{\frac{4s}{d}+1}_{\dot{X}^0(J_0)} + C(E,M)\eta^c\|u\|_{\dot{X}^0(J_0)}.\end{aligned}$$

By taking $\delta, \eta > 0$ small enough, the continuity argument yields

$$\|u\|_{\dot{X}^0(J_0)} \leq 4\delta. \tag{7.11}$$

In particular, (5.5) holds. It remains to check (5.8). Estimating as in (7.8), we get

$$\|e\|_{\dot{N}^0(J_0)} \leq C(E,M)\eta^c\|u\|_{\dot{X}^0(J_0)} \leq C(E,M)\delta\eta^c. \tag{7.12}$$

This shows that (5.8) on J_0 by choosing $\delta, \eta > 0$ small depending only on E and M. Applying Proposition 5.1, we obtain

$$\|u - v\|_{\dot{F}^0(J_0)} \leq \eta^{c/2}. \tag{7.13}$$

Moreover, we also have from Strichartz estimates, (7.4), (7.9) and (7.11) that

$$\begin{aligned}\|u\|_{\dot{F}^s(J_0)} &\lesssim \|u(a)\|_{\dot{H}^s} + \||u|^{\frac{4s}{d}}u\|_{\dot{N}^s(J_0)} + \||u|^{\alpha_2}u\|_{\dot{N}^s(J_0)} \\ &\lesssim C(E) + \|u\|^{\frac{4s}{d}}_{V(J_0)}\|u\|_{\dot{F}^s(J_0)} + C(E,M)\eta^c\|u\|_{\dot{F}^s(J_0)} \\ &\lesssim C(E) + (4\delta)^{\frac{4s}{d}}\|u\|_{\dot{F}^s(J_0)} + C(E,M)\eta^c\|u\|_{\dot{F}^s(J_0)}.\end{aligned} \tag{7.14}$$

By taking $\delta, \eta > 0$ small enough, we get

$$\|u\|_{\dot{F}^s(J_0)} \leq C(E).$$

For $k = 1$, we see that the condition (5.4) is again satisfied by the conservation of mass. By Strichartz estimates, (7.13) implies

$$\|u(t_1) - v(t_1)\|_{L^2} \leq \eta^{c/2}, \tag{7.15}$$

$$\|e^{-i(t-t_1)(-\Delta)^s}(u(t_1) - v(t_1))\|_{\dot{X}^0(J_1)} \lesssim \eta^{c/2}. \tag{7.16}$$

This shows (5.6) and (5.7). By Duhamel's formula, Strichartz estimates, (7.9), (7.10) and (7.16), we have

$$\begin{aligned}\|u\|_{\dot{X}^0(J_1)} &\leq \|e^{-i(t-t_1)(-\Delta)^s}v(t_1)\|_{\dot{X}^0(J_1)} + \|e^{-i(t-t_1)(-\Delta)^s}(u(t_1) - v(t_1))\|_{\dot{X}^0(J_1)} \\ &\quad + C\||u|^{\frac{4s}{d}}u\|_{\dot{N}^0(J_1)} + C\||u|^{\alpha_2}u\|_{\dot{N}^0(J_1)} \\ &\leq 2\delta + \eta^{c/2} + C\|u\|^{\frac{4s}{d}+1}_{\dot{X}^0(J_1)} + C(E,M)\eta^c\|u\|_{\dot{X}^0(J_1)}.\end{aligned}$$

Taking $\delta, \eta > 0$ small enough, the continuity argument implies

$$\|u\|_{\dot{X}^0(J_1)} \leq 4\delta. \tag{7.17}$$

This shows in particular that (5.5) holds. Using (7.17), a similar argument as in (7.9) gives

$$\|e\|_{\dot{N}^0(J_1)} \leq C(E,M)\eta^c\|u\|_{\dot{X}^0(J_1)} \leq C(E,M)\delta\eta^c. \tag{7.18}$$

Choosing $\delta, \eta > 0$ small depending only on E and M, the condition (5.8) holds on J_1. Applying Proposition 5.1, we get

$$\|u - v\|_{\dot{F}^0(J_1)} \leq \eta^{c/4}.$$

By the same argument as in (7.14), we also have

$$\|u\|_{\dot{F}^s(J_1)} \leq C(E).$$

By induction, taking $\delta, \eta > 0$ smaller in each step, we obtain

$$\|u - v\|_{\dot{F}^0(J_k)} \leq \eta^{c/2^{k+1}},$$

and

$$\|u\|_{\dot{F}^s(J_k)} \leq C(E),$$

for each $k = 0, \cdots, L-1$. Adding these estimates over all subintervals J_k which have a nonempty intersection with I_n, we get

$$\|u\|_{\dot{F}^0(I_n)} \leq \|v\|_{\dot{F}^0(I_n)} + \sum_{k=0}^{L-1} \|u-v\|_{\dot{F}^0(J_k)} \leq C(E,M),$$

and

$$\|u\|_{\dot{F}^s(I_n)} \leq \sum_{k=0}^{L-1} \|u\|_{\dot{F}^s(J_k)} \leq C(E,M).$$

Combining these bounds, we prove (7.7). The proof is complete.

(2) The case $\frac{4s}{d} < \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$ and $\mu_1, \mu_2 > 0$. As above, we may assume that $\mu_1 = \mu_2 = 1$. We firstly note that by Theorem 6.1, there exists a unique global solution to (1.2). By Lemma 7.2, the global solution satifies

$$\|u\|_{L^m(\mathbb{R}^+ \times \mathbb{R}^d)} \leq C(E,M),$$

for any $\alpha_1 + 2 + \frac{2}{d-2s} \leq m \leq \alpha_2 + 1 + \frac{2}{d-2s}$. Fix one value of m in this range. Let $\eta > 0$ be a small constant to be chosen later. We divide $\mathbb{R}^+$ into K subintervals $J_k = [t_k, t_{k+1}]$ such that

$$\|u\|_{L^m(J_k \times \mathbb{R}^d)} \sim \eta, \quad \forall k = 0, \cdots, K-1.$$

On each $J_k, k = 0, \cdots, K-1$, u satisfies the integral equation

$$u(t) = e^{-i(t-t_k)(-\Delta)^s} u(t_k) - i \sum_{j=1}^{2} \int_{t_k}^{t} e^{-i(t-\tau)(-\Delta)^s} (|u|^{\alpha_j} u)(\tau) d\tau.$$

By Strichartz estimates and Lemma 7.3, we have

$$\begin{aligned}
\|u\|_{F^s(J_k)} &\lesssim \|u(t_k)\|_{H^s} + \| \langle \nabla \rangle^s (|u|^{\alpha_1} u)\|_{\dot{N}^0(J_k)} + \| \langle \nabla \rangle^s (|u|^{\alpha_2} u)\|_{\dot{N}^0(J_k)} \\
&\lesssim \|u\|_{L^\infty(J_k, H^s)} + \|u\|_{L^m(J_k \times \mathbb{R}^d)}^{\frac{m\epsilon}{2(2+\epsilon)}} \|u\|_{L^\infty(J_k,H^s)}^{a_1(\epsilon)+b_1(\epsilon)} \|u\|_{F^s(J_k)} \\
&\quad + \|u\|_{L^m(J_k \times \mathbb{R}^d)}^{\frac{m\epsilon}{2(2+\epsilon)}} \|u\|_{L^\infty(J_k,H^s)}^{a_2(\epsilon)+b_2(\epsilon)} \|u\|_{F^s(J_k)} \\
&\lesssim \|u\|_{L^\infty(J_k, H^s)} + \eta^{\frac{m\epsilon}{2(2+\epsilon)}} \|u\|_{L^\infty(J_k,H^s)}^{a_1(\epsilon)+b_1(\epsilon)} \|u\|_{F^s(J_k)} + \eta^{\frac{m\epsilon}{2(2+\epsilon)}} \|u\|_{L^\infty(J_k,H^s)}^{a_2(\epsilon)+b_2(\epsilon)} \|u\|_{F^s(J_k)},
\end{aligned}$$

provided that $\epsilon > 0$ is chosen small enough so that (7.4) holds. Here we use the fact that $\left(2+\epsilon, \frac{2d(2+\epsilon)}{d(2+\epsilon)-4s}\right)$ is fractional admissible. By the conservation of mass and energy, we have $\|u\|_{L^\infty(J_k,H^s)} \leq C(E,M)$. Therefore, by choosing $\eta > 0$ small enough depending on E and M, we obtain

$$\|u\|_{F^s(J_k)} \leq C(E,M),$$

for each $k = 0, \cdots, K-1$. Summing these bounds over all subintervals J_k, we prove (7.3).

(3) The case $\frac{4s}{d} \leq \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$, $\mu_1 < 0$, $\mu_2 > 0$ and small mass. Without loss of generality, we may assume that $\mu_1 = -1$ and $\mu_2 = 1$. We will compare the solution of (1.2) with the free fractional Schrödinger equation

$$\left\{ \begin{array}{rcl} i\partial_t \tilde{u} - (-\Delta)^s \tilde{u} &=& 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \tilde{u}(0) &=& u_0. \end{array} \right.$$

By Strichartz estimates, the global solution $\tilde{u}$ obeys the spacetime bounds

$$\|\tilde{u}\|_{\dot{F}^0(\mathbb{R}^+)} \lesssim \|u_0\|_{L^2} \lesssim \sqrt{M}, \tag{7.19}$$

$$\|\tilde{u}\|_{\dot{F}^s(\mathbb{R}^+)} \lesssim \|u_0\|_{\dot{H}^s} \lesssim \sqrt{E}. \tag{7.20}$$

For a spacetime slab $J\times\mathbb{R}^d$, we denote

$$\dot{Y}^0(J):=V(J)\cap W(J),\quad \dot{Y}^s(J):=\left\{u\ :\ |\nabla|^s u\in \dot{Y}^0(J)\right\},\quad Y^s(J)=\dot{Y}^0(J)\cap\dot{Y}^s(J),$$

where the V and W norms are given in (5.1) and (7.5) respectively. Using (7.20), we divide $\mathbb{R}^+$ into K subintervals $J_k=[t_k,t_{k+1}]$ such that

$$\|\tilde{u}\|_{\dot{Y}^s(J_k)}\sim\eta,\quad \forall k=0,\cdots,K-1,$$

where $\eta>0$ is a small constant to be chosen later. By (7.19), we can choose M small enough depending on η so that

$$\|\tilde{u}\|_{Y^s(J_k)}\sim\eta,\quad \forall k=0,\cdots,K-1.$$

We will use the bounds of $\tilde{u}$ to derive the bounds of u on each spacetime slab $J_k\times\mathbb{R}^d$.

For $k=0$, we first deduce from the Duhamel formula and the fact $u(0)=\tilde{u}(0)=u_0$ that

$$u(t)=\tilde{u}(t)+i\int_0^t e^{-i(t-\tau)(-\Delta)^s}|u(\tau)|^{\alpha_1}u(\tau)d\tau-i\int_0^t e^{-i(t-\tau)(-\Delta)^s}|u(\tau)|^{\alpha_2}u(\tau)d\tau.$$

By Strichartz estimates and Lemma 7.4, we have

$$\|u\|_{Y^s(J_0)}\leq\|\tilde{u}\|_{Y^s(J_0)}+C\sum_{j=1}^2\|u\|^{\alpha_j+1}_{Y^s(J_0)}\leq\eta+C\sum_{j=1}^2\|u\|^{\alpha_j+1}_{Y^s(J_0)}.$$

Taking $\eta>0$ small enough, the standard continuity argument yields

$$\|u\|_{Y^s(J_0)}\leq 2\eta.$$

Similarly, by Strichartz estimates and Lemma 7.4,

$$\|u\|_{\dot{Y}^0(J_0)}\leq\|\tilde{u}\|_{\dot{Y}^0(J_0)}+C\sum_{j=1}^2\|u\|^{3-\frac{(d-2s)\alpha_j}{2s}}_{\dot{Y}^0(J_0)}\|u\|^{\frac{d\alpha_j}{2s}-2}_{\dot{Y}^s(J_0)}\lesssim M^{1/2}+\sum_{j=1}^2\eta^{\frac{d\alpha_j}{2s}-2}\|u\|^{3-\frac{(d-2s)\alpha_j}{2s}}_{\dot{Y}^0(J_0)}.$$

Choosing M sufficiently small, the continuity argument shows

$$\|u\|_{\dot{Y}^0(J_0)}\lesssim M^{1/2}.$$

Moreover, Strichartz estimates and Lemma 7.4 again imply that

$$\begin{aligned}\|u-\tilde{u}\|_{\dot{F}^s(J_0)}&\lesssim\sum_{j=1}^2\|u\|^{2-\frac{(d-2s)\alpha_j}{2s}}_{\dot{Y}^0(J_0)}\|u\|^{\frac{d\alpha_j}{2s}-1}_{\dot{Y}^s(J_0)}\\&\lesssim\sum_{j=1}^2 M^{1-\frac{(d-2s)\alpha_j}{4s}}\eta^{\frac{d\alpha_j}{2s}-1}\leq M^\delta,\end{aligned}\tag{7.21}$$

where $\delta>0$ is a small constant provided that $\eta>0$ is taken small enough.

For $k=1$, we use Strichartz estimates together with (7.21) to get

$$\|e^{-i(t-t_1)(-\Delta)^s}(u(t_1)-\tilde{u}(t_1))\|_{\dot{F}^s(J_1)}\lesssim\|u-\tilde{u}\|_{\dot{F}^s(J_0)}\leq M^\delta.\tag{7.22}$$

By Duhamel's formula, Strichartz estimates and the triangle inequality, we have

$$\begin{aligned}\|u\|_{Y^s(J_1)}&\leq\|e^{-i(t-t_1)(-\Delta)^s}u(t_1)\|_{\dot{Y}^0(J_1)}+\|e^{-i(t-t_1)(-\Delta)^s}u(t_1)\|_{\dot{Y}^s(J_1)}+C\sum_{j=1}^2\|u\|^{\alpha_j+1}_{Y^s(J_1)}\\&\leq\|e^{-i(t-t_1)(-\Delta)^s}u(t_1)\|_{\dot{Y}^0(J_1)}+\|e^{-i(t-t_1)(-\Delta)^s}\tilde{u}(t_1)\|_{\dot{Y}^s(J_1)}\\&\quad+\|e^{-i(t-t_1)(-\Delta)^s}(u(t_1)-\tilde{u}(t_1))\|_{\dot{Y}^s(J_1)}+C\sum_{j=1}^2\|u\|^{\alpha_j+1}_{Y^s(J_1)}.\end{aligned}$$

By Duhamel's formula, we see that

$$\begin{aligned} e^{-i(t-t_1)(-\Delta)^s}u(t_1) &= e^{-it(-\Delta)^s}u_0 + i\int_0^{t_1} e^{-i(t-\tau)(-\Delta)^s}|u(\tau)|^{\alpha_1}u(\tau)d\tau \\ &\quad - i\int_0^{t_1} e^{-i(t-\tau)(-\Delta)^s}|u(\tau)|^{\alpha_2}u(\tau)d\tau \\ &= \tilde{u}(t) + \int_0^{t_1} e^{-i(t-\tau)(-\Delta)^s}|u(\tau)|^{\alpha_1}u(\tau)d\tau - i\int_0^{t_1} e^{-i(t-\tau)(-\Delta)^s}|u(\tau)|^{\alpha_2}u(\tau)d\tau, \end{aligned}$$

and $e^{-i(t-t_1)(-\Delta)^s}\tilde{u}(t_1) = e^{-it(-\Delta)^s}u_0 = \tilde{u}(t)$. Therefore,

$$\begin{aligned} \|u\|_{Y^s(J_1)} &\le \|\tilde{u}\|_{\dot{Y}^0(J_1)} + \|\tilde{u}\|_{\dot{Y}^s(J_1)} + \|e^{-i(t-t_1)(-\Delta)^s}(u(t_1)-\tilde{u}(t_1))\|_{\dot{Y}^s(J_1)} + C\sum_{j=1}^2 \|u\|_{Y^s(J_1)}^{\alpha_j+1} \\ &\lesssim M^{1/2} + \eta + M^\delta + C\sum_{j=1}^2 \|u\|_{Y^s(J_1)}^{\alpha_j+1}. \end{aligned}$$

The continuity argument yields

$$\|u\|_{Y^s(J_1)} \le 2\eta, \tag{7.23}$$

provided that η and M are chosen sufficiently small. Similarly,

$$\begin{aligned} \|u\|_{\dot{Y}^0(J_1)} &\le \|e^{-i(t-t_1)(-\Delta)^s}u(t_1)\|_{\dot{Y}^0(J_1)} + C\sum_{j=1}^2 \|u\|_{\dot{Y}^0(J_1)}^{3-\frac{(d-2s)\alpha_j}{2s}} \|u\|_{\dot{Y}^s(J_1)}^{\frac{d\alpha_j}{2s}-2} \\ &\le \|\tilde{u}\|_{\dot{Y}^0(J_1)} + C\sum_{j=1}^2 \|u\|_{\dot{Y}^0(J_1)}^{3-\frac{(d-2s)\alpha_j}{2s}} \|u\|_{\dot{Y}^s(J_1)}^{\frac{d\alpha_j}{2s}-2} \\ &\lesssim M^{1/2} + C\sum_{j=1}^2 \eta^{\frac{d\alpha_j}{2s}-2}\|u\|_{\dot{Y}^0(J_0)}^{3-\frac{(d-2s)\alpha_j}{2s}}. \end{aligned}$$

Choosing M sufficiently small, we get

$$\|u\|_{\dot{Y}^0(J_1)} \lesssim M^{1/2}. \tag{7.24}$$

We also have from Strichartz estimates, (7.22), (7.23) and (7.24) that

$$\begin{aligned} \|u-\tilde{u}\|_{\dot{F}^s(J_1)} &\le \|e^{-i(t-t_1)(-\Delta)^s}(u(t_1)-\tilde{u}(t_1))\|_{\dot{F}^s(J_1)} + C\sum_{j=1}^2 \|u\|_{\dot{Y}^0(J_1)}^{2-\frac{(d-2s)\alpha_j}{2s}} \|u\|_{\dot{F}^s(J_1)}^{\frac{d\alpha_j}{2s}-1} \\ &\lesssim M^\delta + C\sum_{j=1}^2 M^{1-\frac{(d-2s)\alpha_j}{4s}}\eta^{\frac{d\alpha_j}{2s}-1} \lesssim M^\delta, \end{aligned}$$

provided that $\delta > 0$ is chosen sufficiently small.

The same argument applies for the next spacetime slab $J_2 \times \mathbb{R}^d$. By induction, we obtain

$$\|u\|_{Y^s(J_k)} \le 2\eta, \quad \forall k = 0, \cdots, K-1.$$

Summing these bounds over all subintervals J_k, we get $\|u\|_{Y^s(\mathbb{R}^+)} \le C(E)$. By Strichartz estimates and Lemma 7.4, we have

$$\|u\|_{F^s(\mathbb{R}^+)} \lesssim \|u_0\|_{H^s} + \sum_{j=1}^2 \|u\|_{Y^s(\mathbb{R}^+)}^{\alpha_j+1} \le C(E,M) = C(E).$$

This shows (7.3).

7.3. **Global Strichartz bounds imply scattering.** In this subsection, we will use the global Strichartz bounds (7.3) to show the scattering. We first show that $e^{it(-\Delta)^s}u(t)$ has a limit in H^s as $t \to +\infty$. To see this, let $0 < t_1 < t_2$. By Strichartz estimates and Lemma 7.4,

$$\begin{aligned}\|e^{it_2(-\Delta)^s}u(t_2) - e^{it_1(-\Delta)^s}u(t_1)\|_{H^s} &= \Big\| -i\sum_{j=1}^2 \mu_j \int_{t_1}^{t_2} e^{i\tau(-\Delta)^s}|u(\tau)|^{\alpha_j}u(\tau)d\tau\Big\|_{H^s}\\ &\lesssim \Big\| -i\sum_{j=1}^2 \mu_j \int_{t_1}^{t_2} e^{i\tau(-\Delta)^s}|u(\tau)|^{\alpha_j}u(\tau)d\tau\Big\|_{L^\infty([t_1,t_2],H^s)}\\ &\lesssim C(\mu_1,\mu_2)\sum_{j=1}^2 \|u\|^{2-\frac{(d-2s)\alpha_j}{2s}}_{V([t_1,t_2])}\||\nabla|^s u\|^{\frac{d\alpha_j}{2s}-2}_{W([t_1,t_2])}\|\langle\nabla\rangle^s u\|_{V([t_1,t_2])}\\ &\lesssim C(\mu_1,\mu_2)\sum_{j=1}^2 \|u\|^{\alpha_j+1}_{F^s([t_1,t_2])}.\end{aligned}$$

The global Strichartz bounds (7.3) implies that

$$\|e^{it_2(-\Delta)^s}u(t_2) - e^{it_1(-\Delta)^s}u(t_1)\|_{H^s} \to 0,$$

as $t_1, t_2 \to +\infty$. This implies that the limit

$$u_0^+ := \lim_{t\to+\infty} e^{it(-\Delta)^s}u(t)$$

exists in H^s. Moreover,

$$u(t) - e^{-it(-\Delta)^s}u_0^+ = i\sum_{j=1}^2 \mu_j \int_t^{+\infty} e^{-i(t-\tau)(-\Delta)^s}|u(\tau)|^{\alpha_j}u(\tau)d\tau.$$

By the same argument as above, we get

$$\lim_{t\to+\infty}\|u(t) - e^{-it(-\Delta)^s}u_0^+\|_{H^s} = 0.$$

This completes the proof of Theorem 7.1.

8. Blow-up criteria

In this section, we show some criteria for the existence of blow-up H^s and $\dot{H}^{s_c} \cap \dot{H}^s$ solutions for (1.2).

8.1. H^s **blow-up criteria.** By the global well-posedness given in Section 6, the existence of blow-up H^s solutions may only occur for $\mu_2 < 0$ and $\frac{4s}{d} \leq \alpha_2 < \frac{4s}{d-2s}$. When $\frac{4s}{d} < \alpha_2 < \frac{4s}{d-2s}$, the second author in [24] has established some sufficient conditions about existence of blow-up solutions, and derived some sharp thresholds of blow-up and global existence. Here, we investigate the sharp threshold mass of blow-up and global existence for (1.2) with L^2-critical and L^2-subcritical nonlinearities, i.e., $0 < \alpha_1 < \alpha_2 = \frac{4s}{d}$.

Theorem 8.1 (H^s sharp global existence and blow-up criteria)**.** *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $\mu_2 < 0$, $\mu_1 \in \mathbb{R}$ and $0 < \alpha_1 < \alpha_2 = \frac{4s}{d}$. Let $u_0 \in H^s$ be radial and $u \in C([0,T), H^s)$ be the corresponding solution to (1.2). Let Q_2 is the unique (up to symmetries) positive radial solution to the elliptic equation (2.11) with $\alpha = \alpha_2$. Then we have the following sharp criteria for global existence and blow-up of (1.2).*

1. *If $\|u_0\|_{L^2} < \|Q_2\|_{L^2}$, then the solution exists globally in time, i.e. $T = +\infty$.*
2. *If $u_0(x) = c\lambda^{\frac{d}{2}}Q_2(\lambda x)$ for some $c \in \mathbb{C}$ satisfying $|c| \geq 1$ and some $\lambda > 0$, and*
   - *if $\mu_1 > 0$, then the solution either blows up in finite time, i.e. $T < +\infty$ or blows up infinite time, i.e. $T = +\infty$ and satisfies*

$$\|u(t)\|_{\dot{H}^s} \geq Ct^s,$$

   for all $t \geq t_*$ with some constant $C > 0$ and $t_* > 0$ depending only on u_0, s, d.

- *if $\mu_1 < 0$, then the solution either blows up in finite time, i.e. $T < +\infty$ or blows up infinite time, i.e. $T = +\infty$ and there exists a time sequence $(t_n)_{n\geq 1}$ such that $t_n \to +\infty$ and*
$$\|u(t_n)\|_{\dot{H}^s} \to \infty, \tag{8.1}$$
as $n \to \infty$.

Remark 8.2. In [24, Theorem 3.3], the second author proved this result for $\mu_1 > 0$. However, there is an error in the proof given in [24, Theorem 3.3]. Here we extend it to $\mu_1 \in \mathbb{R}$ and give a correct proof.

Proof of Theorem 8.1. (1) We consider separately two cases $\mu_1 > 0$ and $\mu_1 < 0$. In the first case, without loss of generality we take $\mu_1 = 1$ and $\mu_2 = -1$. By the sharp Gagliardo-Nirenberg inequality and the conservation of mass, we have
$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|u(t)\|^2_{\dot{H}^s} + \frac{1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} - \frac{1}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\geq \frac{1}{2}\left(1 - \left(\frac{\|u_0\|_{L^2}}{\|Q_2\|_{L^2}}\right)^{\alpha_2}\right)\|u(t)\|^2_{\dot{H}^s}. \end{aligned}$$
By the assumption $\|u_0\|_{L^2} < \|Q\|_{L^2}$ and the conservation of energy, we see that $\|u(t)\|_{\dot{H}^s}$ is bounded from above for all $t \in [0, T)$. The blow-up alternative implies the solution exists globally in time, i.e. $T = +\infty$. In the second case, we assume $\mu_1 = \mu_2 = -1$. By the sharp Gagliardo-Nirenberg inequality and the conservation of mass,
$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|u(t)\|^2_{\dot{H}^s} - \frac{1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} - \frac{1}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\geq \frac{1}{2}\left(1 - \left(\frac{\|u_0\|_{L^2}}{\|Q_2\|_{L^2}}\right)^{\alpha_2}\right)\|u(t)\|^2_{\dot{H}^s} - \frac{C_1}{\alpha_1+2}\|u_0\|^{\alpha_1+2-\frac{d\alpha_1}{2s}}_{L^2}\|u(t)\|^{\frac{d\alpha_1}{2s}}_{\dot{H}^s}. \end{aligned}$$
We next apply the Young inequality to have for any $\eta > 0$ small enough,
$$E(u(t)) \geq \frac{1}{2}\left(1 - \left(\frac{\|u_0\|_{L^2}}{\|Q_2\|_{L^2}}\right)^{\alpha_2}\right)\|u(t)\|^2_{\dot{H}^s} - \frac{C_1}{\alpha_1+2}\|u_0\|^{\alpha_1+2-\frac{d\alpha_1}{2s}}_{L^2}\left(\eta\|u(t)\|^2_{\dot{H}^s} + \eta^{-\frac{d\alpha_1}{4s-d\alpha_1}}\right). \tag{8.2}$$
Taking $\eta > 0$ small enough depending on $\|u_0\|_{L^2}$ (for instance $\eta = c^2\|u_0\|^{\frac{d\alpha_1}{s}-2\alpha_1-4}_{L^2}$) so that
$$\frac{C_1}{\alpha_1+2}\|u_0\|^{\alpha_1+2-\frac{d\alpha_1}{2s}}_{L^2}\eta \leq \frac{c}{2},$$
for some $0 < c \ll 1$. The conservation of energy and (8.2) imply
$$\frac{1}{2}\left(1 - \left(\frac{\|u_0\|_{L^2}}{\|Q_2\|_{L^2}}\right)^{\alpha_2} - c\right)\|u(t)\|^2_{\dot{H}^s} \leq C(E, M).$$
This shows that $\|u(t)\|_{\dot{H}^s}$ is bounded from above for all $t \in [0, T)$. Therefore the solution exists globally in time. This completes the proof of (1).

(2) Since $u_0(x) = c\lambda^{\frac{d}{2}}Q_2(\lambda x)$, a calculation shows
$$\|u_0\|_{L^2} = |c|\|Q_2\|_{L^2}, \quad \|u_0\|_{\dot{H}^s} = |c|\lambda^s\|Q_2\|_{\dot{H}^s}, \quad \|u_0\|^{\alpha+2}_{L^{\alpha+2}} = |c|^{\alpha+2}\lambda^{\frac{d\alpha}{2}}\|Q_2\|^{\alpha+2}_{L^{\alpha+2}}.$$
Without loss of generality, we assume $\mu_1 \in \{\pm 1\}$ and $\mu_2 = -1$. We have
$$\begin{aligned} E(u_0) &= \frac{1}{2}\|u_0\|^2_{\dot{H}^s} \pm \frac{1}{\alpha_1+2}\|u_0\|^{\alpha_1+2}_{L^{\alpha_1+2}} - \frac{1}{\alpha_2+2}\|u_0\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &= \frac{|c|^2\lambda^{2s}}{2}\|Q_2\|^2_{\dot{H}^s} \pm \frac{|c|^{\alpha_1+2}\lambda^{\frac{d\alpha_1}{2}}}{\alpha_1+2}\|Q_2\|^{\alpha_1+2}_{L^{\alpha_1+2}} - \frac{|c|^{\alpha_2+2}\lambda^{\frac{d\alpha_2}{2}}}{\alpha_2+2}\|Q_2\|^{\alpha_2+2}_{L^{\alpha_2+2}}. \end{aligned}$$
Using the Pohozaev identity $\|Q_2\|^{\alpha_2+2}_{L^{\alpha_2+2}} = \frac{\alpha_2+2}{2}\|Q_2\|^2_{\dot{H}^s}$, we have
$$E(u_0) = -\frac{|c|^2\lambda^{2s}}{2}(|c|^{\alpha_2} - 1)\|Q_2\|^2_{\dot{H}^s} \pm \frac{|c|^{\alpha_1+2}\lambda^{\frac{d\alpha_1}{2}}}{\alpha_1+2}\|Q_2\|^{\alpha_1+2}_{L^{\alpha_1+2}}. \tag{8.3}$$

In the case $\mu_1 = -1$, since $|c| > 1$, it is obvious that $E(u_0) < 0$. In the case $\mu_1 = 1$, we take $\lambda > 0$ such that

$$\frac{2|c|^{\alpha_1}\|Q_2\|^{\alpha_1+2}_{L^{\alpha_1+2}}}{(\alpha_1+2)(|c|^{\alpha_2}-1)\|Q_2\|^2_{\dot{H}^s}} < \lambda^{\frac{4s-d\alpha_1}{2}}.$$

This implies $E(u_0) < 0$. Therefore, in both cases, we have $E(u_0) < 0$.

On the other hand, we apply Lemma 4.4 with $\mu_1 \in \{\pm 1\}$, $\mu_2 = -1$ and $\alpha_2 = \frac{4s}{d}$ to have

$$\begin{aligned}
\frac{d}{dt} M_{\varphi_R}(u(t)) \leq 8s\|u(t)\|^2_{\dot{H}^s} &\pm \frac{4d\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} - \frac{16s}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\
&-2\int_0^\infty m^s \int \left(\psi_{1,R} - C_1(\eta)\psi_{2,R}^{\frac{2}{\alpha_1}}\right)|\nabla u_m(t)|^2 dx dm \\
&-2\int_0^\infty m^s \int \left(\psi_{1,R} - C_2(\eta)\psi_{2,R}^{\frac{d}{2s}}\right)|\nabla u_m(t)|^2 dx dm \\
&+O\Big(R^{-2s} + \eta^{-\beta_1}R^{-\gamma_1} + \eta^{-\beta_2}R^{-\gamma_2} + \eta(1+R^{-2}+R^{-4})\Big) \\
= 16sE(u(t)) &\pm \frac{2(d\alpha_1 - 4s)}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} \\
&-2\int_0^\infty m^s \int \left(\psi_{1,R} - C_1(\eta)\psi_{2,R}^{\frac{2}{\alpha_1}}\right)|\nabla u_m(t)|^2 dx dm \\
&-2\int_0^\infty m^s \int \left(\psi_{1,R} - C_2(\eta)\psi_{2,R}^{\frac{d}{2s}}\right)|\nabla u_m(t)|^2 dx dm \\
&+O\Big(R^{-2s} + \eta^{-\beta_1}R^{-\gamma_1} + \eta^{-\beta_2}R^{-\gamma_2} + \eta(1+R^{-2}+R^{-4})\Big)
\end{aligned}$$

for any $\eta > 0$, where

$$\psi_{1,R} := 2 - \varphi''_R, \quad \psi_{2,R} := 2d - \Delta\varphi_R,$$

and $C_1(\eta), C_2(\eta), \beta_1, \beta_2, \gamma_1, \gamma_2 > 0$. By a similar argument in [3, Appendix], we can choose φ_R and $\eta > 0$ small enough so that

$$\psi_{1,R} - C_1(\eta)\psi_{2,R}^{\frac{2}{\alpha_1}} \geq 0, \quad \psi_{1,R} - C_2(\eta)\psi_{2,R}^{\frac{d}{2s}} \geq 0,$$

for all $r > 0$ and $R > 0$. We next choose $\eta > 0$ small enough and $R > 0$ large enough depending on η, the conservation of energy and the fact $E(u_0) < 0$ imply

$$\frac{d}{dt} M_{\varphi_R}(u(t)) \leq 8sE(u_0) \pm \frac{2(d\alpha_1 - 4s)}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}}, \tag{8.4}$$

for all $t \in [0, T)$.

In the case $\mu_1 = 1$, we obviously have

$$\frac{d}{dt} M_{\varphi_R}(u(t)) \leq 8sE(u_0) < 0,$$

for all $t \in [0, T)$. If $T < +\infty$, then the proof is done. Otherwise, we can take $T = +\infty$. By (8.4), we infer that

$$M_{\varphi_R}(u(t)) \leq -ct, \tag{8.5}$$

for all $t \geq t_1$ with some sufficiently large time $t_1 > 0$ and some constant $c > 0$ depending on s and $E(u_0)$. Moreover, we have from Lemma 4.1 that

$$\begin{aligned}
|M_{\varphi_R}(u(t))| &\leq C(\varphi_R)\left(\||\nabla|^{\frac{1}{2}}u(t)\|^2_{L^2} + \|u(t)\|_{L^2}\||\nabla|^{\frac{1}{2}}u(t)\|_{L^2}\right) \\
&\leq C(\varphi_R)\left(\||\nabla|^{\frac{1}{2}}u(t)\|^2_{L^2} + 1\right) \leq C(\varphi_R)\left(\|u(t)\|^{\frac{1}{s}}_{\dot{H}^s} + 1\right).
\end{aligned} \tag{8.6}$$

Here we use the conservation of mass and the interpolation $\||\nabla|^{\frac{1}{2}}u\|_{L^2} \lesssim \|u\|^{1-\frac{1}{2s}}_{L^2}\|u\|^{\frac{1}{2s}}_{\dot{H}^s}$. Combining (8.5) and (8.6), we obtain

$$\|u(t)\|_{\dot{H}^s} \geq Ct^s,$$

for all $t \geq t_*$ with some constants $C > 0$ and $t_* > 0$ that depend only on u_0, s and d.

In the case $\mu_1 = -1$, we have from (8.4) that

$$\frac{d}{dt} M_{\varphi_R}(u(t)) \leq 8sE(u_0) + \frac{2(4s - d\alpha_1)}{\alpha_1 + 2} \|u(t)\|_{L^{\alpha_1+2}}^{\alpha_1+2}. \tag{8.7}$$

If $T < +\infty$, then the proof is done. Otherwise, $T = +\infty$ and we will show (8.1). Indeed, if (8.1) does not hold, then there exists $C > 0$ such that

$$\sup_{t \in [0,+\infty)} \|u(t)\|_{\dot{H}^s} \leq C. \tag{8.8}$$

Interpolating between L^2 and $L^{\frac{2d}{d-2s}}$ and using the Sobolev embedding, we have

$$\|u(t)\|_{L^{\alpha_1+2}}^{\alpha_1+2} \lesssim \|u_0\|_{L^2}^{\frac{2d-(d-2s)(\alpha_1+2)}{2s}} \|u(t)\|_{\dot{H}^s}^{\frac{d\alpha_1}{2s}} \leq \frac{C}{\alpha_1+2} (|c| \|Q_2\|_{L^2})^{\frac{2d-(d-2s)(\alpha_1+2)}{2s}}.$$

Note that the constant C may change from lines to lines. By (8.3) and (8.7), we get

$$\begin{aligned} \frac{d}{dt} M_{\varphi_R}(u(t)) \leq -4s|c|^2 \lambda^{2s}(|c|^{\alpha_2} - 1)\|Q_2\|_{\dot{H}^s}^2 - \frac{8s|c|^{\alpha_1+2}\lambda^{\frac{d\alpha_1}{2}}}{\alpha_1+2} \|Q_2\|_{L^{\alpha_1+2}}^{\alpha_1+2} \\ + \frac{C}{\alpha_1+2} (|c| \|Q_2\|_{L^2})^{\frac{2d-(d-2s)(\alpha_1+2)}{2s}}. \end{aligned}$$

If we choose $\lambda > 0$ such that

$$\frac{C\|Q_2\|_{L^2}^{\frac{2d-(d-2s)(\alpha_1+2)}{2s}}}{8s|c|^{\frac{d\alpha_1}{2s}} \|Q_2\|_{L^{\alpha_1+2}}^{\alpha_1+2}} < \lambda^{\frac{d\alpha_1}{2}},$$

then

$$\frac{d}{dt} M_{\varphi_R}(u(t)) \leq -\upsilon < 0,$$

for all $t \in [0, +\infty)$ with some constant $\upsilon > 0$. Arguing as in the previous case, we find

$$\|u(t)\|_{\dot{H}^s} \geq Ct^s,$$

for all $t \geq t_*$ with some constant $C > 0$ and $t_* > 0$. This is a contradiction to (8.8). The proof is complete. □

8.2. $\dot{H}^{s_c} \cap \dot{H}^s$ blow-up criteria.

Theorem 8.3 ($\dot{H}^{s_c} \cap \dot{H}^s$ blowup criteria)**.** *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $\mu_2 < 0$, $\frac{4s}{d} < \alpha_2 < \frac{4s}{d-2s}$, $\frac{d\alpha_2 - 4s}{2s} \leq \alpha_1 < \alpha_2$, $\alpha_2 < 4s$ and $u_0 \in \dot{H}^{s_c} \cap \dot{H}^s$ be radial. Suppose the corresponding solution $u \in C([0,T), \dot{H}^{s_c} \cap \dot{H}^s)$ to (1.2) satisfies (4.13). If one of the following conditions holds true:*

1. *$\mu_1 > 0$, $\frac{2s(d\alpha_2 - 4s)}{8s^2 - 4s + (d-2s)\alpha_2} < \alpha_1 < \alpha_2$ and $E(u_0) < 0$;*
2. *$\mu_1 < 0$, $\max\left\{\frac{2s(d\alpha_2 - 4s)}{8s^2 - 4s + (d-2s)\alpha_2}, \frac{4s}{d}\right\} < \alpha_1 < \alpha_2$ and $E(u_0) < 0$;*

then the solution blows up in finite time, i.e. $T < +\infty$.

Proof. Let us start with the following reduction: if $u \in C([0,T), \dot{H}^{s_c} \cap \dot{H}^s)$ is a solution to (1.2) satisfying

$$\frac{d}{dt} M_{\varphi_R}(u(t)) \leq -c\|u(t)\|_{\dot{H}^s}^2, \tag{8.9}$$

$$\|u(t)\|_{\dot{H}^s} \gtrsim 1, \tag{8.10}$$

for all $t \in [0,T)$ with a suitable function φ_R and some small positive constant $c > 0$, then the corresponding solution blows up in finite time, i.e. $T < +\infty$. Indeed, suppose the solution exists globally in time, i.e. $T = +\infty$. By (8.9) and (8.10), we see that $\frac{d}{dt} M_{\varphi_R}(u(t)) \leq -C$ for some $C > 0$. Integrating this bound, we have that $M_{\varphi_R}(u(t))<0$ for all $t\geq t_1$ with some $t_1\gg 1$ large enough. Taking integration over $[t_1,t]$ of (8.9), we obtain

$$M_{\varphi_R}(u(t)) \leq -c\int_{t_1}^t \|u(\tau)\|^2_{\dot{H}^s} d\tau, \tag{8.11}$$

for all $t\geq t_1$. On the other hand, by (4.15) and the assumption (4.13),

$$|M_{\varphi_R}(u(t))| \leq C(\varphi_R)\left(\|u(t)\|^{\frac{1}{s}}_{\dot{H}^s} + \|u(t)\|^{\frac{1}{2s}}_{\dot{H}^s}\right). \tag{8.12}$$

By (8.10) and (8.12), we see that

$$|M_{\varphi_R}(u(t))| \leq C(\varphi_R)\|u(t)\|^{\frac{1}{s}}_{\dot{H}^s}. \tag{8.13}$$

We thus get from (8.11) and (8.13) that

$$M_{\varphi_R}(u(t)) \leq -C(\varphi_R)\int_{t_1}^t |M_{\varphi_R}(u(\tau))|^{2s} d\tau,$$

for all $t\geq t_1$. By nonlinear integral inequality, it yields that $M_{\varphi_R}(u(t)) \lesssim -C(\varphi_R)|t-t_*|^{1-2s}$ for some finite $t_*<+\infty$. This shows that $M_{\varphi_R}(u(t))\to-\infty$ as $t\uparrow t_*$. Therefore the solution cannot exist for all time $t\geq 0$ and consequencely we must have $T<+\infty$.

We now prove (8.9) and (8.10) under the hypotheses of Theorem 8.3. The second condition (8.10) follows easily from the fact $E(u_0)<0$. In fact, suppose it is not true. Then there exists a sequence $(t_k)_k\subset[0,+\infty)$ such that $\|u(t_k)\|_{\dot{H}^s}\to 0$ as $k\to\infty$. Thanks to the Gagliardo-Nirenberg inequality given in Lemma 2.5 and the assumption (4.13), we see that $\|u(t_k)\|^{\alpha_j+2}_{L^{\alpha_j+2}}\to 0$ as $k\to\infty$. We thus get $E(u(t_k))\to 0$, which is a contradiction to $E(u(t_k))=E(u_0)<0$. Let us show (8.9).

(1) The case $\mu_1>0$, $\frac{2s(d\alpha_2-4s)}{8s^2-4s+(d-2s)\alpha_2}<\alpha_1<\alpha_2$ and $E(u_0)<0$: Without loss of generality, we assume $\mu_1=1,\mu_2=-1$. Applying Lemma 4.5, we get for any $\eta>0$,

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(u(t)) &\leq 8s\|u(t)\|^2_{\dot{H}^s} + \frac{4d\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} - \frac{4d\alpha_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\quad +O\left(R^{-2(s-s_c)}+C_1(\eta)R^{-\vartheta}+C_2(\eta)R^{-2(s-s_c)}+\eta\|u(t)\|^2_{\dot{H}^s}\right) \\ &= 4d\alpha_2E(u(t)) - 2(d\alpha_2-4s)\|u(t)\|^2_{\dot{H}^s} - \frac{4d(\alpha_2-\alpha_1)}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} \\ &\quad +O\left(R^{-2(s-s_c)}+C_1(\eta)R^{-\vartheta}+C_2(\eta)R^{-2(s-s_c)}+\eta\|u(t)\|^2_{\dot{H}^s}\right).\end{aligned}$$

Using the conservation of energy, the assumption $E(u_0)<0$ and the fact $d\alpha_2>4s$, the condition (8.9) holds with $c=d\alpha_2-4s$ by taking $\eta>0$ small enough and $R>0$ large enough depending on η.

(2) The case $\mu_1<0$, $\max\left\{\frac{2s(d\alpha_2-4s)}{8s^2-4s+(d-2s)\alpha_2},\frac{4s}{d}\right\}<\alpha_1<\alpha_2$ and $E(u_0)<0$: We assume $\mu_1=\mu_2=-1$. By Lemma 4.5, we have for any $\eta>0$,

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}(u(t)) &\leq 8s\|u(t)\|^2_{\dot{H}^s} - \frac{4d\alpha_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} - \frac{4d\alpha_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\quad +O\left(R^{-2(s-s_c)}+C_1(\eta)R^{-\vartheta}+C_2(\eta)R^{-2(s-s_c)}+\eta\|u(t)\|^2_{\dot{H}^s}\right) \\ &= 4d\alpha_1E(u(t)) - 2(d\alpha_1-4s)\|u(t)\|^2_{\dot{H}^s} - \frac{4d(\alpha_2-\alpha_1)}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\quad +O\left(R^{-2(s-s_c)}+C_1(\eta)R^{-\vartheta}+C_2(\eta)R^{-2(s-s_c)}+\eta\|u(t)\|^2_{\dot{H}^s}\right).\end{aligned}$$

By the conservation of energy, the assumption $E(u_0)<0$ and the fact $d\alpha_1>4s$, we see that (8.9) holds with $c=d\alpha_1-4s$ provided that $\eta>0$ is taken small enough and $R>0$ is taken large enough depending on η. The proof is complete. $\square$

We end this section by giving some criteria for global existence of $\dot{H}^{s_c}\cap\dot{H}^s$ solutions to (1.2).

Lemma 8.4. *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $\mu_1 \in \mathbb{R}$, $\mu_2 < 0$, $\frac{4s}{d} < \alpha_2 < \frac{4s}{d-2s}$ and $\frac{d\alpha_2-4s}{2s} \leq \alpha_1 < \alpha_2$. Let $u_0 \in \dot{H}^{s_c} \cap \dot{H}^s$ be radial and the corresponding solution u to (1.2) defined on the maximal time interval $[0,T)$. If*

$$\sup_{t\in[0,T)} \|u(t)\|_{\dot{H}^{s_c}} < S_{\text{gs}}, \tag{8.14}$$

where S_{gs} is given in (2.19), then the solution exists globally in time, i.e. $T = +\infty$.

Proof. We have

$$E(u(t)) = \frac{1}{2}\|u(t)\|^2_{\dot{H}^s} + \frac{\mu_1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} + \frac{\mu_2}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}}.$$

In the case $\mu_1 > 0$ and $\mu_2 < 0$ (WLG we assume $\mu_1 = 1$ and $\mu_2 = -1$), we use the sharp Gagliardo-Nirenberg inequality (2.21) to have

$$\begin{aligned} E(u(t)) &\geq \frac{1}{2}\|u(t)\|^2_{\dot{H}^s} - \frac{1}{\alpha_2+2}\|u(t)\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &\geq \frac{1}{2}\left(1 - \left(\frac{\|u(t)\|_{\dot{H}^{s_c}}}{S_{\text{gs}}}\right)^{\alpha_2}\right)\|u(t)\|^2_{\dot{H}^s}. \end{aligned}$$

Thanks to the conservation of energy and the assumption (8.14), we obtain $\|u(t)\|_{\dot{H}^s} < \infty$ for all $t \in [0,T)$. The blow-up alternative implies the solution exists globally in time.

In the case $\mu_1 < 0$ and $\mu_2 < 0$ (we assume $\mu_1 = \mu_2 = -1$), we have

$$E(u(t)) + \frac{1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} \geq \frac{1}{2}\left(1 - \left(\frac{\|u(t)\|_{\dot{H}^{s_c}}}{S_{\text{gs}}}\right)^{\alpha_2}\right)\|u(t)\|^2_{\dot{H}^s}.$$

If $\frac{d\alpha_2-4s}{2s} = \alpha_1$ or $\alpha_1 + 2 = \frac{d\alpha_2}{2s} = \alpha_c$ (see (1.4) for the notation), we use the Sobolev embedding $\dot{H}^{s_c} \hookrightarrow L^{\alpha_c}$ to have

$$\frac{1}{2}\left(1 - \left(\frac{\|u(t)\|_{\dot{H}^{s_c}}}{S_{\text{gs}}}\right)^{\alpha_2}\right)\|u(t)\|^2_{\dot{H}^s} \leq E(u(t)) + \frac{1}{\alpha_1+2}\|u(t)\|^{\alpha_1+2}_{\dot{H}^{s_c}}.$$

The conservation of energy and (8.14) then imply $\|u(t)\|_{\dot{H}^s} < \infty$ for all $t \in [0,T)$. This shows $T = +\infty$.

If $\frac{d\alpha_2-4s}{2s} < \alpha_1 < \alpha_2$, then $\alpha_c < \alpha_1 + 2 < \alpha_2 + 2 < \frac{2d}{d-2s}$. Then interpolation implies

$$\begin{aligned} \|u(t)\|^{\alpha_1+2}_{L^{\alpha_1+2}} &\lesssim \|u(t)\|^{\theta(\alpha_1+2)}_{L^{\alpha_c}}\|u(t)\|^{(1-\theta)(\alpha_1+2)}_{L^{\frac{2d}{d-2s}}} \\ &\lesssim \|u(t)\|^{\theta(\alpha_1+2)}_{\dot{H}^{s_c}}\|u(t)\|^{(1-\theta)(\alpha_1+2)}_{\dot{H}^s}, \end{aligned}$$

for some $\theta \in (0,1)$. It is easy to check that $(1-\theta)(\alpha_1+2) < 2$. We then apply the Young inequality to get for any $\eta > 0$,

$$\|u(t)\|^{(1-\theta)(\alpha_1+2)}_{\dot{H}^s} \lesssim \eta\|u(t)\|^2_{\dot{H}^s} + C(\eta).$$

This yields

$$\frac{1}{2}\left(1 - \left(\frac{\|u(t)\|_{\dot{H}^{s_c}}}{S_{\text{gs}}}\right)^{\alpha_2}\right)\|u(t)\|^2_{\dot{H}^s} \leq E(u_0) + \frac{C}{\alpha_1+2}\|u(t)\|^{\theta(\alpha_1+2)}_{\dot{H}^{s_c}}\left(\eta\|u(t)\|^2_{\dot{H}^s} + C(\eta)\right).$$

Using (8.14) and taking $\eta > 0$ small enough, we see that $\|u(t)\|_{\dot{H}^s} < \infty$ for all $t \in [0,T)$. This again implies $T = +\infty$. The proof is complete. $\square$

Lemma 8.5. *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $\mu_1 \in \mathbb{R}$, $\mu_2 < 0$, $\frac{4s}{d} < \alpha_2 < \frac{4s}{d-2s}$ and $\frac{d\alpha_2-4s}{2s} \leq \alpha_1 < \alpha_2$. Let $u_0 \in \dot{H}^{s_c} \cap \dot{H}^s$ be radial and the corresponding solution u to (1.2) defined on the maximal time interval $[0,T)$. If*

$$S_{\text{gs}} \leq \sup_{t\in[0,T)} \|u(t)\|_{\dot{H}^{s_c}} < \infty, \quad \sup_{t\in[0,T)} \|u(t)\|_{L^{\alpha_c}} < L_{\text{gs}}, \tag{8.15}$$

where S_{gs} and L_{gs} are given in (2.19) and (2.20) respectively, then the solution exists globally in time, i.e. $T = +\infty$.

Proof. The proof is similar to the one of Lemma 8.4 by using (2.22). We omit the details. □

Lemma 8.6. *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $\mu_1 \in \mathbb{R}$, $\mu_2 > 0$, $\frac{4s}{d} < \alpha_2 < \frac{4s}{d-2s}$, $\frac{d\alpha_2 - 4s}{2s} \leq \alpha_1 < \alpha_2$ and $u_0 \in \dot{H}^{s_c} \cap \dot{H}^s$ be radial. Let $u \in C([0,T), \dot{H}^{s_c} \cap \dot{H}^s)$ be the corresponding solution u to (1.2) satisfying (4.13). Then the solution exists globally in time, i.e. $T = +\infty$.*

Proof. The proof is again similar to the one of Lemma 8.4. We omit the details. □

9. Blow-up dynamics

9.1. **Blow-up dynamics in the mass-critical case.** In this subsection, we study dynamical properties of blow-up H^s solutions for (1.2) with $\mu_1 \in \mathbb{R}$, $\mu_2 < 0$ and $0 < \alpha_1 < \alpha_2 = \frac{4s}{d}$. Note that in this setting, the existence of blow-up H^s solutions has been established in Theorem 8.1. Using the compactness lemma given in Lemma 2.7, we obtain the following L^2-concentration of blow-up H^s solutions to (1.2).

Theorem 9.1 (L^2-concentration)**.** *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $\mu_1 \in \mathbb{R}$, $\mu_2 < 0$, $0 < \alpha_1 < \alpha_2 = \frac{4s}{d}$ and $u_0 \in H^s$ be radial. Assume that the corresponding solution u to (1.2) blows up in finite time $0 < T < +\infty$. Let a be a real valued non-negative function defined on $[0,T)$ satisfying*

$$a(t)\|u(t)\|_{\dot{H}^s}^{\frac{1}{s}} \to \infty, \tag{9.1}$$

as $t \uparrow T$. Then there exists $x(t) \in \mathbb{R}^d$ such that

$$\liminf_{t\uparrow T} \int_{|x-x(t)|\leq a(t)} |u(t,x)|^s dx \geq \int |Q(x)|^2 dx, \tag{9.2}$$

where Q is the unique (up to symmetries) positive radial solution to (2.11) with $\alpha = \frac{4s}{d}$.

Remark 9.2.
- In [24, Theorem 4.2], the author proved this result for $\mu_1 > 0$. Here we extend it to $\mu_1 \in \mathbb{R}$.
- This result shows that the L^2-norm of blow-up solutions must concentrate by an amount which is bounded from below by $\|Q\|_{L^2}$ at the blow-up time. Moreover, as mentioned in [24], the rate of L^2-concentration of blow-up solutions is $\|u(t)\|_{\dot{H}^s}^{-\frac{1}{s}+\delta}$ with $0 < \delta < \frac{1}{s}$.

Proof of Theorem 9.1. Without loss of generality, we may assume $\mu_1 \in \{\pm 1\}$ and $\mu_2 = -1$. Let $(t_n)_{n\geq 1}$ be a sequence such that $t_n \uparrow T$. Set

$$\lambda_n := \left(\frac{\|Q\|_{\dot{H}^s}}{\|u(t_n)\|_{\dot{H}^s}}\right)^{\frac{1}{s}}, \quad v_n(x) := \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n x).$$

By the blow-up alternative, we have $\|u(t_n)\|_{\dot{H}^s} \to \infty$ as $n \to \infty$, so $\lambda_n \to 0$ as $n \to \infty$. Moreover,

$$\|v_n\|_{L^2} = \|u(t_n)\|_{L^2} = \|u_0\|_{L^2}, \tag{9.3}$$

$$\|v_n\|_{\dot{H}^s} = \lambda_n^s \|u(t_n)\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s}. \tag{9.4}$$

On the other hand,

$$\begin{aligned} H(v_n) &:= \frac{1}{2}\|v_n\|_{\dot{H}^s}^2 - \frac{d}{2d+4s}\|v_n\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2} \\ &= \lambda_n^{2s}\left(\frac{1}{2}\|u(t_n)\|_{\dot{H}^s}^2 - \frac{d}{2d+4s}\|u(t_n)\|_{L^{\frac{4s}{d}+2}}^{\frac{4s}{d}+2}\right) \\ &= \lambda_n^{2s}\left(E(u(t_n)) - \frac{\mu_1}{\alpha_1+2}\|u(t_n)\|_{L^{\alpha_1+2}}^{\alpha_1+2}\right) \\ &= \frac{\|Q\|_{\dot{H}^s}^2}{\|u(t_n)\|_{\dot{H}^s}^2}\left(E(u_0) - \frac{\mu_1}{\alpha_1+2}\|u(t_n)\|_{L^{\alpha_1+2}}^{\alpha_1+2}\right). \end{aligned}$$

Applying the sharp Gagliardo-Nirenberg inequality and using the conservation of mass, we obtain

$$\begin{aligned}|H(v_n)| &\leq \frac{\|Q\|^2_{\dot{H}^s}}{\|u(t_n)\|^2_{\dot{H}^s}}\left(|E(u_0)| + \frac{1}{\alpha_1+2}\|u(t_n)\|^{\alpha_1+2}_{L^{\alpha_1+2}}\right) \\ &\leq \frac{\|Q\|^2_{\dot{H}^s}}{\|u(t_n)\|^2_{\dot{H}^s}}\left(|E(u_0)| + \frac{C_1}{\alpha_1+2}\|u_0\|^{\alpha_1+2-\frac{d\alpha_1}{2s}}_{L^2}\|u(t_n)\|^{\frac{d\alpha_1}{2s}}_{\dot{H}^s}\right).\end{aligned}$$

Since $\frac{d\alpha_1}{2s} < 2$ and $\|u(t_n)\|_{\dot{H}^s} \to \infty$ as $n \to \infty$, we learn that $|H(v_n)| \to 0$ as $n \to \infty$. From this and the fact $\|v_n\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s}$, we have

$$\|v_n\|^{\frac{4s}{d}+2}_{L^{\frac{4s}{d}+2}} \to \frac{d+2s}{d}\|Q\|^2_{\dot{H}^s}, \tag{9.5}$$

as $n \to \infty$. The sequence $(v_n)_{n\geq 1}$ satisfies the assumptions of Lemma 2.7 with

$$m^{\frac{4s}{d}+2} = \frac{d+2s}{d}\|Q\|^2_{\dot{H}^s}, \quad M^2 = \|Q\|^2_{\dot{H}^s}.$$

Thus, there exists a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^d$ such that up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } H^s,$$

as $n \to \infty$ with $\|V\|_{L^2} \geq \|Q\|_{L^2}$. In particular,

$$v(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup V \text{ weakly in } L^2.$$

We thus have for every $R > 0$,

$$\liminf_{n\to\infty} \int_{|x|\leq R} \lambda_n^d |u(t_n, \lambda x + x_n)|^2 dx \geq \int_{|x|\leq R} |V(x)|^2 dx.$$

By a change of variables,

$$\liminf_{n\to\infty} \int_{|x-x_n|\leq R\lambda_n} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |V(x)|^2 dx.$$

By (9.1), we also have

$$\frac{a(t_n)}{\lambda_n} = \frac{a(t_n)\|u(t_n)\|^{\frac{1}{s}}_{\dot{H}^s}}{\|Q\|^{\frac{1}{s}}_{\dot{H}^s}} \to \infty,$$

as $n \to \infty$. We thus get

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq a(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |V(x)|^2 dx,$$

for every $R > 0$, which implies that

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq a(t_n)} |u(t_n, x)|^2 dx \geq \int |V(x)|^2 dx \geq \int |Q(x)|^2 dx.$$

Since $(t_n)_{n\geq 1}$ is arbitrary, we infer that

$$\liminf_{t\uparrow T} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq a(t)} |u(t, x)|^2 dx \geq \int |Q(x)|^2 dx. \tag{9.6}$$

Observe that for every $t \in [0, T)$, the function $y \mapsto \int_{|x-y|\leq a(t)} |u(t,x)|^2 dx$ is continuous and tends to zero as $|y|$ tends to infinity. Therefore, there exists a function $x(t) \in \mathbb{R}^d$ such that

$$\sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq a(t)} |u(t, x)|^2 dx = \int_{|x-x(t)|\leq a(t)} |u(t, x)|^2 dx.$$

This combined with (9.6) show (9.2). The proof is complete. □

In order to study the limiting profile of blow-up H^s solutions with minimal mass $\|Q\|_{L^2}$, we need the following characterization of the ground state.

Lemma 9.3 (Characterization of ground state [20])**.** *Let $d \geq 1$ and $0 < s < 1$. If $u \in H^s$ is such that $\|u\|_{L^2} = \|Q\|_{L^2}$ and*

$$H(u) := \frac{1}{2}\|u\|^2_{\dot{H}^s} - \frac{d}{2d+4s}\|u\|^{\frac{4s}{d}+2}_{L^{\frac{4s}{d}+2}} = 0,$$

then u is of the form

$$u(x) = e^{i\theta}\lambda^{\frac{d}{2}} Q(\lambda x + x_0),$$

for some $\theta \in \mathbb{R}, \lambda > 0$ and $x_0 \in \mathbb{R}^d$.

Theorem 9.4 (Limiting profile)**.** *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$, $\mu_1 \in \mathbb{R}$, $\mu_2 < 0$ and $0 < \alpha_1 < \alpha_2 = \frac{4s}{d}$. Let $u_0 \in H^s$ be radial be such that $\|u_0\|_{L^2} = \|Q\|_{L^2}$, where Q is the unique (up to symmetries) positive radial solution to (2.11) with $\alpha = \frac{4s}{d}$. Assume that the corresponding solution u to (1.2) blows up in finite time $0 < T < +\infty$. Then there exist $\theta(t) \in \mathbb{R}^d$, $\lambda(t) > 0$ and $x(t) \in \mathbb{R}^d$ such that*

$$e^{i\theta(t)}\lambda^{\frac{d}{2}}(t) u(t, \lambda(t)\cdot + x(t)) \to Q \text{ strongly in } H^s,$$

as $t \uparrow T$.

Proof. We use the notations given in the proof of Theorem 9.1. We see that

$$v(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup V \text{ weakly in } H^s,$$

as $n \to \infty$ with $\|V\|_{L^2} \geq \|Q\|_{L^2}$. By the semi-continuity of weak convergence and (9.3), we have

$$\|Q\|_{L^2} \leq \|V\|_{L^2} \leq \liminf_{n\to\infty} \|v_n\|_{L^2} = \|Q\|_{L^2}.$$

This shows that

$$\|V\|_{L^2} = \|Q\|_{L^2} = \lim_{n\to\infty} \|v_n\|_{L^2}. \tag{9.7}$$

In particular, $v_n(\cdot + x_n) \to V$ strongly in L^2 as $n \to \infty$. On the other hand, by the Gagliardo-Nirenberg inequality, we have

$$v_n(\cdot + x_n) \to V \text{ strongly in } L^{\frac{4s}{d}+2},$$

as $n \to \infty$. Indeed, by (9.4),

$$\begin{aligned}\|v_n(\cdot + x_n) - V\|^{\frac{4s}{d}+2}_{L^{\frac{4s}{d}+2}} &\lesssim \|v_n(\cdot + x_n) - V\|^{\frac{4s}{d}}_{L^2} \|v_n(\cdot + x_n) - V\|^2_{\dot{H}^s} \\ &\lesssim \left(\|Q\|_{\dot{H}^s} + \|V\|_{\dot{H}^s}\right)^2 \|v_n(\cdot + x_n) - V\|^{\frac{4s}{d}}_{L^2} \to 0,\end{aligned}$$

as $n \to \infty$. Moreover, using (9.5) and (9.7), the sharp Gagliardo-Nirenberg inequality implies

$$\|Q\|^2_{\dot{H}^s} = \frac{d}{d+2s}\lim_{n\to\infty}\|v_n\|^{\frac{4s}{d}+2}_{L^{\frac{4s}{d}+2}} = \frac{d}{d+2s}\|V\|^{\frac{4s}{d}+2}_{L^{\frac{4s}{d}+2}} \leq \left(\frac{\|V\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4s}{d}} \|V\|^2_{\dot{H}^s} = \|V\|^2_{\dot{H}^s}.$$

Using this together with (9.4), the semi-continuity of weak convergence implies

$$\|V\|_{\dot{H}^s} \leq \liminf_{n\to\infty} \|v_n\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s} \leq \|V\|_{\dot{H}^s}.$$

Thus,

$$\|V\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s} = \lim_{n\to\infty}\|v_n\|_{\dot{H}^s}.$$

This and (9.7) imply $\|V\|_{H^s} = \|Q\|_{H^s} = \lim_{n\to\infty}\|v_n\|_{H^s}$. Hence, $v_n(\cdot + x_n) \to V$ strongly in H^s as $n \to \infty$. In particular, we have

$$H(V) = \lim_{n\to\infty} H(v_n) = 0.$$

This shows that there exists $V \in H^s$ such that

$$\|V\|_{L^2} = \|Q\|_{L^2}, \quad H(V) = 0.$$

The characterization of ground state given in Lemma 9.3 implies $V(x) = e^{i\theta}\lambda^{\frac{d}{2}}Q(\lambda x+x_0)$ for some $\theta \in \mathbb{R}$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$. We thus conclude that

$$v_n(\cdot + x_n) = \lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot +x_n) \to V = e^{i\theta}\lambda^{\frac{d}{2}}Q(\lambda \cdot +x_0) \text{ strongly in } H^s,$$

as $n \to \infty$. Redefining variables $\tilde{\theta}_n := -\theta$, $\tilde{\lambda}_n := \lambda_n\lambda^{-1}$ and $\tilde{x}_n := \lambda_n\lambda^{-1}x_0 + x_n$, we obtain

$$e^{i\tilde{\theta}_n}\tilde{\lambda}_n^{\frac{d}{2}} u(t_n, \tilde{\lambda}_n \cdot +\tilde{x}_n) \to Q \text{ strongly in } H^s,$$

as $n \to \infty$. Since $(t_n)_{n\geq 1}$ is arbitrary, we infer that there exist $\theta(t) \in \mathbb{R}$, $\lambda(t) > 0$ and $x(t) \in \mathbb{R}^d$ such that

$$e^{i\theta(t)}\lambda^{\frac{d}{2}}(t)u(t, \lambda(t) \cdot +x(t)) \to Q \text{ strongly in } H^s,$$

as $t \uparrow T$. The proof is complete. □

9.2. **Blow-up dynamics in the mass-supercritical case.**

Theorem 9.5 (Blow-up concentration). *Let $d \geq 2$, $\frac{d}{2d-1} \leq s < 1$ and $\mu_1 \in \mathbb{R}$, $\mu_2 < 0$, $\frac{4s}{d} < \alpha_2 < \frac{4s}{d-2s}$, $\frac{d\alpha_2 - 4s}{2s} \leq \alpha_1 < \alpha_2$ and $u_0 \in \dot{H}^{s_c} \cap \dot{H}^s$ be radial. Assume that the corresponding solution u to (1.2) blows up in finite time $0 < T < +\infty$ and satisfies (4.13). Let a be a real valued non-negative function defined on $[0, T)$ satisfying*

$$a(t)\|u(t)\|_{\dot{H}^s}^{\frac{1}{s-s_c}} \to \infty, \tag{9.8}$$

as $t \uparrow T$. Then there exist $y(t), z(t) \in \mathbb{R}^d$ such that

$$\liminf_{t\uparrow T} \int_{|y-y(t)|\leq a(t)} |(-\Delta)^{\frac{s_c}{2}} u(t, y)|^2 dy \geq S_{\text{gs}}^2, \tag{9.9}$$

and

$$\liminf_{t\uparrow T} \int_{|z-z(t)|\leq a(t)} |u(t, z)|^{\alpha_c} dz \geq L_{\text{gs}}^{\alpha_c}. \tag{9.10}$$

Remark 9.6. In [23], the second author proved a similar result for the nonlinear Schrödinger equation with combined power-type nonlinearities. Here we extend his result in the context of the fractional nonlinear Schrödinger equation. Note that since the uniqueness (up to symmetries) of solutions to (2.14) and (2.17) are not yet known, we need to introduce the notions of Sobolev and Lebesgue ground states (see Definition 2.6).

Proof of Theorem 9.5. We assume without loss of generality that $\mu_1 \in \{\pm 1\}$ and $\mu_2 = -1$. Let $(t_n)_{n\geq 1}$ be a sequence such that $t_n \uparrow T$ and $g \in \mathcal{G}$. Denote

$$\lambda_n := \left(\frac{\|g\|_{\dot{H}^s}}{\|u(t_n)\|_{\dot{H}^s}}\right)^{\frac{1}{s-s_c}}, \quad v_n(y) := \lambda_n^{\frac{2s}{\alpha_2}} u(t_n, \lambda_n y).$$

Thanks to the assumption (4.13), the blow-up alternative implies that $\lambda_n \to 0$ as $n \to \infty$. A direct computation shows

$$\|v_n\|_{\dot{H}^{s_c}} = \|u(t_n)\|_{\dot{H}^{s_c}} < \infty, \tag{9.11}$$

uniformly in n and

$$\|v_n\|_{\dot{H}^s} = \lambda_n^{s-s_c}\|u(t_n)\|_{\dot{H}^s} = \|g\|_{\dot{H}^s}. \tag{9.12}$$

Moreover,

$$\begin{aligned}H(v_n) &:= \frac{1}{2}\|v_n\|^2_{\dot{H}^s} - \frac{1}{\alpha_2+2}\|v_n\|^{\alpha_2+2}_{L^{\alpha_2+2}} \\ &= \lambda_n^{2(s-s_c)}\left(\frac{1}{2}\|u(t_n)\|^2_{\dot{H}^s} - \frac{1}{\alpha_2+2}\|u(t_n)\|^{\alpha_2+2}_{L^{\alpha_2+2}}\right) \\ &= \lambda_n^{2(s-s_c)}\left(E(u(t_n)) - \frac{\mu_1}{\alpha_1+2}\|u(t_n)\|^{\alpha_1+2}_{L^{\alpha_1+2}}\right) \\ &= \frac{\|g\|^2_{\dot{H}^s}}{\|u(t_n)\|^2_{\dot{H}^s}}\left(E(u_0) - \frac{\mu_1}{\alpha_1+2}\|u(t_n)\|^{\alpha_1+2}_{L^{\alpha_1+2}}\right). \end{aligned} \tag{9.13}$$

We next claim that $H(v_n) \to 0$ as $n \to \infty$, which implies

$$\|v_n\|^{\alpha_2+2}_{L^{\alpha_2+2}} \to \frac{\alpha_2+2}{2}\|g\|^2_{\dot{H}^s}, \tag{9.14}$$

as $n \to \infty$. In fact, if $\frac{d\alpha_2-4s}{2s} = \alpha_1$, then $\alpha_1+2 = \frac{d\alpha_2}{2s} = \alpha_c$ (see (1.4)). Therefore, the Sobolev embedding $\dot{H}^{s_c} \hookrightarrow L^{\alpha_c}$ and the assumption (4.13) imply

$$\|u(t_n)\|^{\alpha_1+2}_{L^{\alpha_1+2}} = \|u(t_n)\|^{\alpha_c}_{L^{\alpha_c}} \lesssim \|u(t_n)\|^{\alpha_c}_{\dot{H}^{s_c}} < \infty.$$

This together with $\|u(t_n)\|_{\dot{H}^s} \to \infty$ as $n \to 0$ yield $H(v_n) \to 0$ as $n \to \infty$. If $\frac{d\alpha_2-4s}{2s} < \alpha_1 < \alpha_2 < \frac{4s}{d-2s}$ or $\alpha_c < \alpha_1+2 < \frac{2d}{d-2s}$, then the interpolation estimate implies

$$\begin{aligned}\|u(t_n)\|^{\alpha_1+2}_{L^{\alpha_1+2}} &\lesssim \|u(t_n)\|^{\theta(\alpha_1+2)}_{L^{\alpha_c}}\|u(t_n)\|^{(1-\theta)(\alpha_1+2)}_{L^{\frac{2d}{d-2s}}} \\ &\lesssim \|u(t_n)\|^{\theta(\alpha_1+2)}_{\dot{H}^{s_c}}\|u(t_n)\|^{(1-\theta)(\alpha_1+2)}_{\dot{H}^s},\end{aligned}$$

where $\theta := \frac{2d\alpha_2-(d-2s)(\alpha_1+2)\alpha_2}{(4s-(d-2s)\alpha_2)(\alpha_1+2)} \in (0,1)$. It is easy to check that $(1-\theta)(\alpha_1+2) < 2$. This combined with (9.13) show $H(v_n) \to 0$ as $n \to \infty$. Combining two cases, we prove the claim. Using (9.12) and (9.14), we see that the sequence $(v_n)_{n\geq 1}$ satisfies the conditions of Lemma 2.9 with

$$m^{\alpha_2+2} = \frac{\alpha_2+2}{2}\|g\|^2_{\dot{H}^s}, \quad M^2 = \|g\|^2_{\dot{H}^s}.$$

Thus, there exists a sequence $(y_n)_{n\geq 1}$ in $\mathbb{R}^d$ such that up to a subsequence,

$$v_n(\cdot + y_n) = \lambda_n^{\frac{2s}{\alpha_2}} u(t_n, \lambda_n \cdot + y_n) \rightharpoonup P \text{ weakly in } \dot{H}^{s_c} \cap \dot{H}^s,$$

as $n \to \infty$ with $\|P\|_{\dot{H}^{s_c}} \geq S_{\text{gs}}$. In particular, we have

$$(-\Delta)^{\frac{s_c}{2}} v(\cdot + y_n) = \lambda_n^{\frac{d}{2}}[(-\Delta)^{\frac{s_c}{2}} u](t_n, \lambda_n \cdot + y_n) \rightharpoonup (-\Delta)^{\frac{s_c}{2}} P \text{ weakly in } L^2.$$

Thus, for any $R > 0$,

$$\liminf_{n\to\infty} \int_{|y|\leq R} \lambda_n^d |[(-\Delta)^{\frac{s_c}{2}} u](t_n, \lambda_n y + y_n)|^2 dy \geq \int_{|y|\leq R} |(-\Delta)^{\frac{s_c}{2}} P(y)|^2 dy,$$

or

$$\liminf_{n\to\infty} \int_{|y-y_n|\leq R\lambda_n} |[(-\Delta)^{\frac{s_c}{2}} u](t_n, y)|^2 dy \geq \int_{|y|\leq R} |(-\Delta)^{\frac{s_c}{2}} P(y)|^2 dy.$$

Since

$$\frac{a(t_n)}{\lambda_n} = \frac{a(t_n)\|u(t_n)\|^{\frac{1}{s-s_c}}_{\dot{H}^s}}{\|g\|^{\frac{1}{s-s_c}}_{\dot{H}^s}} \to \infty,$$

as $n \to \infty$, we have

$$\liminf_{n\to\infty} \sup_{z\in\mathbb{R}^d} \int_{|y-z|\leq a(t_n)} |(-\Delta)^{\frac{s_c}{2}} u(t_n, y)|^2 dy \geq \int_{|y|\leq R} |(-\Delta)^{\frac{s_c}{2}} P(y)|^2 dy,$$

for every $R > 0$, which means that

$$\liminf_{n\to\infty} \sup_{z\in\mathbb{R}^d} \int_{|y-z|\leq a(t_n)} |(-\Delta)^{\frac{s_c}{2}} u(t_n, y)|^2 dy \geq \int |(-\Delta)^{\frac{s_c}{2}} P(y)|^2 dy \geq S^2_{\text{gs}}.$$

Since $(t_n)_{n\geq 1}$ is arbitrary, we infer that

$$\liminf_{t\uparrow T} \sup_{z\in\mathbb{R}^d} \int_{|y-z|\leq a(t)} |(-\Delta)^{\frac{s_c}{2}} u(t,y)|^2 dy \geq S_{\rm gs}^2.$$

Note that for every $t\in[0,T)$, the function $z\mapsto \int_{|y-z|\leq a(t)} |(-\Delta)^{\frac{s_c}{2}} u(t,y)|^2 dy$ is continuous and tends to zero as $|z|$ tends to infinity. As a result, we get

$$\sup_{z\in\mathbb{R}^d} \int_{|y-z|\leq a(t)} |(-\Delta)^{\frac{s_c}{2}} u(t,y)|^2 dy = \int_{|y-y(t)|\leq a(t)} |(-\Delta)^{\frac{s_c}{2}} u(t,y)|^2 dy,$$

for some $y(t)\in\mathbb{R}^d$. This proves (9.9). The proof of (9.10) is similar using the second item of Lemma 2.9. The proof is complete. □

Let us now study the limiting profile of blow-up $\dot H^{s_c}\cap \dot H^s$ solutions with critical norms. To do so, we recall the following characterization of ground states.

Lemma 9.7 (Characterization of ground states [21]). *Let $d\geq 2$, $0<s<1$ and $\frac{4s}{d}<\alpha<\frac{4s}{d-2s}$.*

(1) *If $u\in \dot H^{s_c}\cap\dot H^s$ is such that $\|u\|_{\dot H^{s_c}} = S_{\rm gs}$ and*
$$H(u) := \frac{1}{2}\|u\|^2_{\dot H^s} - \frac{1}{\alpha_2+2}\|u\|^{\alpha_2+2}_{L^{\alpha_2+2}} = 0,$$
then u is of the form
$$u(y) = e^{i\theta}\lambda^{\frac{2s}{\alpha_2}} g(\lambda y + y_0),$$
for some $g\in\mathcal{G}$, $\theta\in\mathbb{R}, \lambda>0$ and $y_0\in\mathbb{R}^d$.

(2) *If $u\in L^{\alpha_c}\cap \dot H^s$ is such that $\|u\|_{L^{\alpha_c}} = L_{\rm gs}$ and*
$$H(u) := \frac{1}{2}\|u\|^2_{\dot H^s} - \frac{1}{\alpha_2+2}\|u\|^{\alpha_2+2}_{L^{\alpha_2+2}} = 0,$$
then u is of the form
$$u(z) = e^{i\vartheta}\rho^{\frac{2s}{\alpha_2}} k(\rho z + z_0),$$
for some $k\in\mathcal{K}$, $\vartheta\in\mathbb{R}, \rho>0$ and $z_0\in\mathbb{R}^d$.

Theorem 9.8 (Limiting profile with critical norms). *Let $d\geq 2$, $\frac{d}{2d-1}\leq s<1$ and $\mu_1\in\mathbb{R}$, $\mu_1<0$, $\frac{4s}{d}<\alpha_2<\frac{4s}{d-2s}$ and $\frac{d\alpha_2-4s}{2s}\leq\alpha_1<\alpha_2$. Let $u_0\in\dot H^{s_c}\cap\dot H^s$ be radial such that the corresponding solution u to (1.2) blows up at finite time $0<T<+\infty$.*

(1) *Assume that*
$$\sup_{t\in[0,T)} \|u(t)\|_{\dot H^{s_c}} = S_{\rm gs}. \tag{9.15}$$
Then there exists $g\in\mathcal{G}$, $\theta(t)\in\mathbb{R}, \lambda(t)>0$ and $y(t)\in\mathbb{R}^d$ such that
$$e^{i\theta(t)}\lambda^{\frac{2s}{\alpha_2}}(t) u(t,\lambda(t)\cdot + y(t)) \to g \text{ strongly in } \dot H^{s_c}\cap\dot H^s,$$
as $t\uparrow T$.

(2) *Assume that*
$$\sup_{t\in[0,T)} \|u(t)\|_{\dot H^{s_c}} <\infty, \quad \sup_{t\in[0,T)} \|u(t)\|_{L^{\alpha_c}} = L_{\rm gs}. \tag{9.16}$$
Then there exist $k\in\mathcal{K}, \vartheta(t)\in\mathbb{R}, \rho(t)>0$ and $z(t)\in\mathbb{R}^d$ such that
$$e^{i\vartheta(t)}\rho^{\frac{2s}{\alpha_2}}(t) u(t,\rho(t)\cdot + z(t)) \to k \text{ strongly in } L^{\alpha_c}\cap\dot H^s,$$
as $t\uparrow T$.

Proof. We only treat the first term, the second one is similar. It is enough to show that for any $(t_n)_{n\geq 1}$ satisfying $t_n \uparrow T$, there exist a subsequence still denoted by $(t_n)_{n\geq 1}$, $\tilde{g} \in \mathcal{G}$, sequences $\theta_n \in \mathbb{R}, \lambda_n > 0$ and $y_n \in \mathbb{R}^d$ such that

$$e^{it\theta_n} \lambda_n^{\frac{2s}{\alpha_2}} u(t_n, \lambda_n \cdot + y_n) \to \tilde{g} \text{ strongly in } \dot{H}^{s_c} \cap \dot{H}^s, \tag{9.17}$$

as $n \to \infty$. Using the notation given in the proof of Theorem 9.5, we have

$$v_n(\cdot + y_n) = \lambda_n^{\frac{2s}{\alpha_2}} u(t_n, \lambda_n \cdot + y_n) \rightharpoonup P \text{ weakly in } \dot{H}^{s_c} \cap \dot{H}^s,$$

as $n \to \infty$ with $\|P\|_{\dot{H}^{s_c}} \geq S_{\mathrm{gs}}$. By the semi-continuity of weak convergence, (9.11) and (9.15),

$$S_{\mathrm{gs}} \leq \|P\|_{\dot{H}^{s_c}} \leq \liminf_{n\to\infty} \|v_n\|_{\dot{H}^{s_c}} \leq S_{\mathrm{gs}}.$$

We thus get

$$S_{\mathrm{gs}} = \|P\|_{\dot{H}^{s_c}} = \lim_{n\to\infty} \|v_n\|_{\dot{H}^{s_c}}. \tag{9.18}$$

This shows that $v_n(\cdot + y_n) \to P$ strongly in $\dot{H}^{s_c}$ as $n \to \infty$. Using the sharp Gagliardo-Nirenberg inequality (2.13), we have

$$v_n(\cdot + y_n) \to P \text{ strongly in } L^{\alpha_2+2},$$

as $n \to \infty$. Indeed,

$$\begin{aligned}
\|v_n(\cdot + y_n) - P\|_{L^{\alpha_2+2}}^{\alpha_2+2} &\lesssim \|v_n(\cdot + y_n) - P\|_{\dot{H}^{s_c}}^{\alpha_2} \|v_n(\cdot + y_n) - P\|_{\dot{H}^s}^2 \\
&\lesssim \left(\|g\|_{\dot{H}^s} + \|P\|_{\dot{H}^s}\right)^2 \|v_n(\cdot + y_n) - P\|_{\dot{H}^{s_c}}^{\alpha_2} \to 0,
\end{aligned}$$

as $n \to \infty$. Using (9.14) and (9.15), the sharp Gagliardo-Nirenberg inequality (2.21) yields

$$\|g\|_{\dot{H}^s}^2 = \frac{2}{\alpha_2+2} \lim_{n\to\infty} \|v_n\|_{L^{\alpha_2+2}}^{\alpha_2+2} = \frac{2}{\alpha_2+2} \|P\|_{L^{\alpha_2+2}}^{\alpha_2+2} \leq \left(\frac{\|P\|_{\dot{H}^{s_c}}}{S_{\mathrm{gs}}}\right)^{\alpha_2} \|P\|_{\dot{H}^s}^2 = \|P\|_{\dot{H}^s}^2.$$

This combined with

$$\|P\|_{\dot{H}^s} \leq \liminf_{n\to\infty} \|v_n\|_{\dot{H}^s} \leq \|g\|_{\dot{H}^s}$$

show

$$\|P\|_{\dot{H}^s} = \|g\|_{\dot{H}^s} = \lim_{n\to\infty} \|v_n\|_{\dot{H}^s}. \tag{9.19}$$

Combining (9.18), (9.19) and the fact $v(\cdot + y_n) \rightharpoonup P$ weakly in $\dot{H}^{s_c} \cap \dot{H}^s$, we conclude that

$$v_n(\cdot + y_n) \to P \text{ strongly in } \dot{H}^{s_c} \cap \dot{H}^s,$$

as $n \to \infty$. In particular, we have

$$H(P) = \lim_{n\to\infty} H(v_n) = 0.$$

Therefore, there exists $P \in \dot{H}^{s_c} \cap \dot{H}^s$ such that

$$\|P\|_{\dot{H}^{s_c}} = S_{\mathrm{gs}}, \quad H(P) = 0.$$

Thanks to the characterization of ground states given in Lemma 9.7, we have $P(y) = e^{i\theta} \lambda^{\frac{2s}{\alpha_2}} \tilde{g}(\lambda y + y_0)$ for some $\theta \in \mathbb{R}, \lambda > 0$ and $y_0 \in \mathbb{R}^d$. We thus obtain

$$v_n(\cdot + y_n) = \lambda_n^{\frac{2s}{\alpha_2}} u(t_n, \lambda_n \cdot + y_n) \to P = e^{i\theta} \lambda^{\frac{2s}{\alpha_2}} \tilde{g}(\lambda \cdot + y_0) \text{ strongly in } \dot{H}^{s_c} \cap \dot{H}^s,$$

as $n \to \infty$. Redefining variables, we prove (9.17). The proof is complete. ∎

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References

[1] **J. Bergh, J. Löfstöm**, *Interpolation spaces*, Springer, New York, 1976. 4

[2] **S. Bhattarai**, *On fractional Schrödinger systems of Choquard type*, J. Differential Equations 263 (2017), No. 6, doi: 10.1016/j.jde.2017.04.034 3197–3229. 1, 2

[3] **T. Boulenger, D. Himmelsbach, E. Lenzmann** *Blowup for fractional NLS*, J. Funct. Anal. 271 (2016), No. 9, doi: 10.1016/j.jfa.2016.08.011, 2569–2603. 1, 3, 4, 7, 10, 11, 12, 13, 15, 30

[4] **T. Cazenave, F. B. Weissler**, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. 14 (1990), 807–836.

[5] **T. Cazenave**, *Semilinear Schrödinger equations*, Courant Lectures Notes in Mathematics 10, New York, AMS, 2003.

[6] **X. Cheng, C. Miao, L. Zhao**, *Global well-posedness and scattering for nonlinear Schrödinger equations with combined nonlinearities in the radial case*, J. Differential Equations 261 (2016), No. 6, doi: 10.1016/j.jde.2016.04.031, 2881–2934. 2

[7] **Y. Cho**, *Short-range scattering of Hartree type fractional NLS*, J. Differential Equations 262 (2017), No. 1, doi: 10.1016/j.jde.2016.09.025, 116–144. 1

[8] **Y. Cho, H. Hajaiej, G. Hwang, T. Ozawa**, *On the Cauchy problem of fractional Schrödinger equations with Hartree type nonlimearity*, Funkcial. Ekvac. 56 (2013), No. 2, doi: 10.1619/fesi.56.193, 193–224. 1

[9] **Y. Cho, G. Hwang, S. Kwon, S. Lee**, *Profile decompositions and blow-up phenomena of mass critical fractional Schrödinger equations*, Nonlinear Anal. 86 (2013), doi: 10.1016/j.na.2013.03.002, 12–29. 1

[10] **Y. Cho, G. Hwang, S. Kwon, S. Lee**, *On finite time blow-up for the mass-critical Hartree equations*, Proc. Roy. Soc. Edinburgh Sect. A 145 (2015), No. 3, doi: 10.1017/S030821051300142X, 467–479. 1

[11] **Y. Cho, G. Hwang, S. Kwon, S. Lee**, *Well-posedness and ill-posedness for the cubic fractional Schrödinger equations*, Discrete Contin. Dyn. Syst. 35 (2015), No. 7, doi: 10.3934/dcds.2015.35.2863, 2863–2880. 1

[12] **Y. Cho, G. Hwang, S. Kwon, S. Lee**, *Profile decompositions of fractional Schrödinger equations with angularly regular data*, J. Differential Equations 256 (2014), No. 8, doi: 10.1016/j.jde.2014.01.030, 3011–3037. 1

[13] **Y. Cho, G. Hwang, T. Ozawa**, *On the focusing energy-critical fractional nonlinear Schrödinger equations*, Adv. Differential Equations 23 (2018), No. 3-4, 161–192. 1

[14] **Y. Cho, S. Lee**, *Strichartz estimates in spherical coordinates*, Indiana Univ. Math. J. 62 (2013), No. 3, 991–1020. 1, 5

[15] **Y. Cho, T. Ozawa**, *Sobolev inequalities with symmetry*, Commun. Contemp. Math. 11 (2009), No. 3, doi: 10.1142/S0219199709003399, 355–365. 1, 13

[16] **Y. Cho, T. Ozawa, S. Xia**, *Remarks on some dispersive estimates*, Commun. Pure Appl. Anal. 10 (2011), No. 4, doi: 10.3934/cpaa.2011.10.1121, 1121–21128. 1, 5

[17] **M. Christ, I. Weinstein**, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. 100 (1991), No. 1, 87–109.

[18] **J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao**, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$*, Comm. Pure Appl. Math. 57 (2004), No. 8, doi: 0.1002/cpa.20029, 987–1014. 20

[19] **V. D. Dinh**, *Well-posedness of nonlinear fractional Schrödinger and wave equations in Sobolev spaces*, Int. J. Appl. Math. 31 (2018), No. 4, doi: 10.12732/ijam.v31i4.1, 483–525. 1, 5, 9

[20] **V. D. Dinh**, *On blowup solutions to the focusing mass-critical nonlinear fractional Schrödinger equation*, Commun. Pure Appl. Anal. 18 (2019), No. 2, doi: 10.3934/cpaa.2019034, 689–708. 1, 2, 6, 8, 9, 10, 36

[21] **V. D. Dinh**, *A study on blowup solutions to the focusing L^2-supercritical nonlinear fractional Schrödinger equation*, J. Math. Phys. 59 (2018), doi: 10.1063/1.5027713, 071506. 1, 2, 6, 7, 8, 10, 15, 39

[22] **R. L. Frank, E. Lenzmann**, *Uniqueness of nonlinear ground states for fractional Laplacians in $\mathbb{R}$*, Acta Math. 210 (2013), No. 2, doi: 10.1007/s11511-013-0095-9, 261–318. 1, 7

[23] **B. Feng**, *On the blow-up solutions for the nonlinear Schrödinger equation with combined power-type nonlinearities*, J. Evol. Equ. 18 (2018), No. 1, doi: 10.1007/s00028-017-0397-z, 203–220. 2, 37

[24] **B. Feng**, *On the blow-up solutions for the fractional nonlinear Schrödinger equation with combined power-type nonlinearities*, Commun. Pure Appl. Anal. 17 (2018), No. 5, doi: 10.3934/cpaa.2018085, 1785–1804. 1, 3, 4, 8, 28, 29, 34

[25] **B. Feng, H. Zhang**, *Stability of standing waves for the fractional Schrödinger-Hartree equation*, J. Math. Anal. Appl. 460 (2018), No. 1, doi: 10.1016/j.jmaa.2017.11.060, 352–364. 1, 2

[26] **B. Feng, H. Zhang**, *Stability of standing waves for the fractional Schrödinger-Choquard equation*, Comput. Math. Appl. 75 (2018), No. 7, doi: 10.1016/j.camwa.2017.12.025, 2499–2507. 1, 2

[27] **R. L. Frank, E. Lenzmann, L. Silvestre**, *Uniqueness of radial solutions for the fractional Laplacian*, Comm. Pure Appl. Math. 69 (2016), No. 9, doi: 10.1002/cpa.21591, 1671–1725. 7

[28] **J. Fröhlich, G. Jonsson, E. Lenzmann**, *Boson stars as solitary waves*, Comm. Math. Phys. 274 (2007), No.1, doi: 10.1007/s00220-007-0272-9, 1–30. 1

[29] **J. Ginibre, G. Velo**, *The global Cauchy problem for the nonlinear Klein-Gordon equation*, Math. Z. 189 (1985), 487–505. 4

[30] **Z. Guo, Y. Sire, Y. Wang, L. Zhao**, *On the energy-critical fractional Schrödinger equation in the radial case*, preprint arXiv:1310.8616, 2013. 1, 2

[31] **Z. Guo, Y. Wang**, *Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations*, J. Anal. Math. 124 (2014), No. 1, doi: 10.1007/s11854-014-0025-6, 1–38. 5

[32] **Q. Guo, S. Zhu**, *Sharp threshold of blow-up and scattering for the fractional Hartree equation*, J. Differential Equations 264 (2018), No. 4, doi: 10.1016/j.jde.2017.11.001, 2802–2832. 1

[33] **Q. Guo, S. Zhu**, *Sharp criteria of scattering for the fractional NLS*, preprint arXiv:1706.02549, 2017. 2

[34] **Y. Hong, Y. Sire**, *On fractional Schrödinger equations in Sobolev spaces*, Commun. Pure Appl. Anal. 14 (2015), No. 6, doi: 10.3934/cpaa.2015.14.2265, 2265–2282. 1

[35] **Y. Ke**, *Remark on the Strichartz estimates in the radial case*, J. Math. Anal. Appl. 387 (2012), No. 2, doi: 10.1016/j.jmaa.2011.09.039, 857–861. 1, 5

[36] **K. Kirkpatrick, E. Lenzmann, G. Staffilani**, *On the continuum limit for discrete NLS with long-range lattice interactions*, Comm. Math. Phys. 317 (2013), No. 3, doi: 10.1007/s00220-012-1621-x, 563–591. 1

[37] **N, Laskin**, *Fractional Quantum Mechanics and Lèvy Path Integrals*, Physics Letter A 268 (2000) 298–304. 1

[38] **N, Laskin**, *Fractional Schrödinger equations*, Physics Review E 66 (2002) 056108. 1

[39] **F. Merle, P. Raphaël**, *Blow up of critical norm for some radial L^2 super critical nonlinear Schrödinger equations*, Amer. J. Math. 130 (2008), No. 4, doi: 10.1353/ajm.0.0012, 945–978. 15

[40] **C. Miao, G. Xu, L. Zhao**, *The dynamics of the 3D radial NLS with combined terms*, Comm. Math. Phys. 318 (2013), No. 3, doi: 10.1007/s00220-013-1677-2, 767–808. 2

[41] **C. Miao, T. Zhao, J. Zheng**, *On the 4D nonlinear Schrödinger equation with combined terms under the energy threshold*, Calc. Var. Partial Differential Equations 56 (2017), doi: 10.1007/s00526-017-1264-z, 56–179. 2

[42] **C.Peng, Q. Shi**, *Stability of standing waves for the fractional nonlinear Schrödinger equation*, J. Math. Phys. 59 (2018), doi: 10.1063/1.5021689, 011508. 2

[43] **C. Sun, H. Wang, X. Yao, J. Zheng**, *Scattering below ground state of focusing fractional nonlinear Schrödinger equation with radial data*, Discrete Contin. Dyn. Syst. 38 (2018), No. 4, doi: 10.3934/dcds.2018091, 2207–2228. 2

[44] **T. Tao, M. Visan, X. Zhang**, *The nonlinear Schrödinger equation with combined power-type nonlinearities*, Comm. Partial Differential Equations 32 (2007), No. 8, doi: 10.1080/03605300701588805, 1281–1343. 2, 3, 4, 18

[45] **H. Triebel**, *Theory of function spaces*, Basel Birkhäuser, 1983. 4

[46] **G. X. Xu, J. W. Yang**, *Long time dynamics of the 3D radial NLS with the combined terms*, Acta Math. Sin. (Engl. Ser.) 32 (2016), No. 5, doi: 10.1007/s10114-016-5401-y, 521–540. 2

[47] **J. Zhang, S. Zhu**, *Stability of standing waves for the nonlinear fractional Schrödinger equation*, J. Dynam. Differential Equations 29 (2017), No. 3, doi: 10.1007/s10884-015-9477-3, 1017–1030. 1

[48] **S. Zhu**, *On the blow-up solutions for the nonlinear fractional Schrödinger equation*, J. Differential Equations 261 (2016), No. 2, doi: 10.1016/j.jde.2016.04.007, 1506–1531. 1, 6

[49] **S. Zhu**, *Existence of stable standing waves for the fractional Schrödinger equations with combined nonlinearities*, J. Evol. Equ. 17 (2017), No. 3, doi: 10.1007/s00028-016-0363-1, 1003–1021. 1, 2, 3, 7

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